SCHOOL OF MATHEMATICS AND STATISTICS        Spring Semester 2012-2013

Applicable Analysis                                          2 hours 30 minutes

Answer four questions. If you answer more than four questions, only your best four will be counted.

You may use the following results when answering questions on this paper.

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^n e^{bt} (\alpha &gt; -1)$</td>
<td>$\Gamma(\alpha + 1) \frac{(s - b)^{\alpha + 1}}{(s - b)^{\alpha + 1}}$</td>
</tr>
<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$f(t) e^{bt}$</td>
<td>$F(s - b)$</td>
</tr>
<tr>
<td>$f^{(n)}(t)$</td>
<td>$s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0) s^{n-k}$</td>
</tr>
<tr>
<td>$tf(t)$</td>
<td>$-F'(s)$</td>
</tr>
</tbody>
</table>
(i) Define what is meant by the statement that \( \int_a^\infty f(x) \, dx \) exists.

\[ (2 \text{ marks}) \]

Prove, from your definition, each of the following statements:

(a) \( \int_0^\infty \frac{1}{(x+1)(x+2)} \, dx \) exists;

(b) \( \int_e^\infty \frac{1}{x \ln x} \, dx \) does not exist.

\[ (6 \text{ marks}) \]

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form \( \int_a^\infty f(x) \, dx \). Your statement should include conditions under which the results are valid.

\[ (4 \text{ marks}) \]

Prove each of the following, stating any standard results you need to use:

(a) \( \int_e^\infty \frac{2 + \cos x}{x \ln x} \, dx \) diverges;

(b) \( \int_0^\infty \frac{1}{1 + x \sqrt{x}} \, dx \) converges.

\[ (7 \text{ marks}) \]

(iii) Decide whether each of the following integrals converges or diverges and prove your assertions.

(a) \( \int_0^1 \frac{dx}{\sqrt{1-x}} \);

(b) \( \int_0^1 \frac{dx}{(\cos x) \sqrt{1-x}} \).

\[ (6 \text{ marks}) \]
State, without proof, the theorem concerning differentiation of an integral of the form $\int_a^\infty f(x, y) \, dx$. Your statement should include conditions under which the result holds. (4 marks)

Show that $\int_0^\infty e^{-x^2} \sin(xy) \, dx$ converges for all $y \in \mathbb{R}$.

Let $c > 0$. Prove that the function $F$ defined on $[-c, c]$ by

$$F(y) = \int_0^\infty e^{-x^2} \sin(xy) \, dx \quad (-c \leq y \leq c)$$

is differentiable on $[-c, c]$ and that

$$2F'(y) = 1 - y F(y)$$

for $-c \leq y \leq c$. (10 marks)

Deduce that (*) holds for all $y \in \mathbb{R}$. (2 marks)

(ii) Define the $\Gamma$ function. (2 marks)

Prove that

(a) $\int_0^\infty x^8 e^{-x^2} \, dx = \frac{105 \sqrt{\pi}}{32}$;

(b) $\int_0^1 (\ln x)^9 \, dx = -(9!)$. (7 marks)
Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. \( (3 \text{ marks}) \)

Prove that

\[
B(x, y) = 2 \int_0^{\pi/2} \cos^{2x-1}\theta \sin^{2y-1}\theta \, d\theta \quad (x > 0, y > 0)
\]

and

\[
B(x, y) = \int_0^\infty \frac{u^{x-1}}{(1+u)^{x+y}} \, du \quad (x > 0, y > 0).
\] \( (4 \text{ marks}) \)

Prove each of the following, stating any standard results you need to use:

(a) \[ \int_0^{\pi/2} \sin^2 \theta \sqrt{\tan \theta} \, d\theta = \frac{3\pi\sqrt{2}}{8}; \]

(b) \[ \int_0^\infty \frac{t^4}{(1+t^4)^2} \, dt = \frac{\pi\sqrt{2}}{16}. \]

(c) the area enclosed by the curve \(|x|^6 + |y|^3 = 1\) is

\[
\frac{4}{9\sqrt{\pi}} \Gamma(\frac{1}{6}) \Gamma(\frac{1}{3}) \] \( (18 \text{ marks}) \)
4  (i) Define what is meant by the statement that \( \int_{0}^{\infty} f(t)e^{-st} \, dt \) has abscissa of convergence \( c \).

\( (2 \text{ marks}) \)

Find the abscissa of convergence of \( \int_{0}^{\infty} te^{-st} \, dt \), giving reasons for your answer.

\( (4 \text{ marks}) \)

(ii) In each of the following cases, find the function continuous on \( [0, \infty) \), with the given Laplace transform:

(a) \( \frac{2}{(s + 1)(s + 3)} \) (\( s > -1 \));

(b) \( \frac{1}{s^2 + 2s + 2} \) (\( s > -1 \));

(c) \( \frac{2}{(s + 2)(s^2 + 2s + 2)} \) (\( s > -1 \)).

\( (8 \text{ marks}) \)

(iii) Let \( b > 0 \). Using Beta and Gamma functions show that

\[ \int_{0}^{\infty} \frac{x^3}{(x^6 + b^2)} \, dx = \frac{\pi}{3\sqrt{3} b^{2/3}}. \]

\( (7 \text{ marks}) \)

By considering

\[ \int_{0}^{\infty} \sin(x^2 t) \, dx, \]

prove that

\[ \int_{0}^{\infty} \sin x^3 \, dx = \frac{\pi}{3\sqrt{3} \Gamma\left(\frac{3}{4}\right)}. \]

\( (4 \text{ marks}) \)
5. Let \( f : [0, \infty) \to \mathbb{R} \) be continuous and suppose that the Laplace transform \( F = L(f) \) exists on \((c, \infty)\) for some \( c \in \mathbb{R} \). State, without proof, the formula giving \( L\left(\frac{f(t)}{t}\right) \) in terms of \( F \). Your statement should include sufficient conditions to ensure the validity of the formula. \( (2 \text{ marks}) \)

Prove that the Laplace transform of \( \frac{\cos t - e^{-t}}{t} \) is given by the formula

\[
L\left(\frac{\cos t - e^{-t}}{t}\right) = -\frac{1}{2} \ln (1 + s^2) + \ln (1 + s).
\]

\( (7 \text{ marks}) \)

Verify that, for \( s \neq -1 \),

\[
\frac{2}{(s^2 + 1)(s + 1)^2} = \frac{1}{(s + 1)^2} + \frac{1}{(s + 1)} - \frac{s}{s^2 + 1}
\]

\( (1 \text{ mark}) \)

Hence find the solution of the differential equation

\[
t \frac{d^2y}{dt^2} + 2 \frac{dy}{dt} + ty = -2te^{-t} \quad (t \geq 0)
\]

such that \( y(0) = 0 \). \( (15 \text{ marks}) \)

End of Question Paper