The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2012-2013**

Mathematics (Numerical Methods and Vector Spaces) **2 hours**

*Answer **four** questions. If you answer more than four questions, only your best four will be counted.*

1.

Let $$A = \begin{pmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}.$$

(i) Find the LU decomposition of A, where L is a lower triangular matrix with ones on the principal diagonal and U is an upper triangular matrix. **(*6 marks*)**

(ii) Verify that L^{-1} and U^{-1} have, respectively, the forms:

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}, \quad U^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{1}{15} & d \\ 0 & \frac{4}{15} & \frac{1}{14} \\ 0 & 0 & \frac{15}{56} \end{pmatrix}$$

and find the values of a, b, c and d. **(*6 marks*)**

(iii) Explain how you would use the result of part (ii) to find A^{-1}. Given that it has the form:

$$A^{-1} = \frac{1}{56}\begin{pmatrix} 15 & 4 & 1 \\ 4 & e & 4 \\ 1 & 4 & 15 \end{pmatrix},$$

find the value of e. **(*3 marks*)**

(iv) The Crank-Nicolson scheme for finding the approximate solution of the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

where $u(0,t) = 14, u(1,t) = 0$ and $u(x,0) = 21 - 28\left|x - \frac{1}{4}\right|$, is

$$-ru_{i-1,j+1} + 2(1+r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + 2(1-r)u_{i,j} + ru_{i+1,j} + O(k^3, kh^2)$$

where $r = k/h^2$ and $u_{i,j} = u(ih, jk)$. Letting $h = 0.25, k = 0.0625$, set up a table showing the values of u at the grid points for $t = 0$, and hence calculate the values of u at the grid points for $t = 0.0625$. **(*10 marks*)**

2.

(i) Let $A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -2 & 3 \end{pmatrix}$, $B = \begin{pmatrix} 7 & 3 & 1 \\ 3 & 9 & 3 \\ 2 & 6 & 8 \end{pmatrix}$.

Evaluate AB and hence or otherwise find A^{-1}.

(*4 marks*)

(ii) The solution of the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is to be approximated in the square region:

$$\left\{(x, y) : 0 \le x \le \frac{3}{2}, 0 \le y \le \frac{3}{2}\right\},$$

subject to the boundary conditions:

$$u = 3 + 3y - 2y^2 \text{ when } x = 0,$$
$$u = 3 + x - 2x^2 \text{ when } y = 0 \text{ or } y = \frac{3}{2},$$
$$\frac{\partial u}{\partial x} = -5 \text{ when } x = \frac{3}{2}.$$

Taking $h = k = \frac{1}{2}$, draw up a suitable grid for a numerical analysis of the problem, and mark on it known values of u. Also indicate on your diagram an appropriate notation for the unknown values and any fictitious values you will require to use.

(*6 marks*)

(iii) Write down equations relating the variables specified in part (ii) and, by eliminating any fictitious values and making use of any symmetry in the diagram, show that they are of the form $A\mathbf{x} = \mathbf{b}$ where A is the matrix in part (i) and $\mathbf{u}$ and $\mathbf{b}$ are column vectors. Hence solve the equations for the unknown values of $\mathbf{u}$.

(*15 marks*)

3.

(i) Give a brief description of a cubic spline, including a clear statement of the conditions that must be satisfied by the two formulae on either side of a datum point. **(*6 marks*)**

(ii) By deriving appropriate formulae from the requirements detailed in part (i), find the cubic spline which fits the following set of data:

x	0	1	2	3
$f(x)$	15	35	25	40

subject to the additional requirement that the tangent at each end should be horizontal. **(*19 marks*)**

You may use without proof that the inverse of the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \quad \text{is} \quad \frac{1}{45}\begin{pmatrix} 26 & -7 & 2 & -1 \\ -7 & 14 & -4 & 2 \\ 2 & -4 & 14 & -7 \\ -1 & 2 & -7 & 26 \end{pmatrix}.$$

4.

(i) Write down the conditions that must be satisfied by $\{f_i\}_{i=1}^N$ if it forms a basis for the Hilbert space V. Write down an expression for the vector f in V in terms of this basis.

What are the additional conditions that must be satisfied by $\{f_i\}_{i=1}^N$ if it forms an orthonormal basis for V? With these conditions, show that the coordinates, c_i, of f can be evaluated using $c_i = (f, f_i)$ and that the inner product of f and g, a vector in V with coordinates d_i, is given by

$$(f,g) = \sum_{i=1}^{N} c_i d_i^* .$$

(*12 marks*)

(ii) Find the norms of the vectors $f_1 = (1,2,0)$ and $f_2 = (-2,1,1)$ in $\mathbb{C}^3$. Show that they are orthogonal and hence construct an orthonormal basis for $\mathbb{C}^3$ using f_1 and f_2. Find the coordinates of the vectors $f = (\mathrm{j},1,1)$ and $g = (2\mathrm{j},0,\mathrm{j})$ in this basis and hence verify the inner product formula in part (i).

(*13 marks)*

5.

(i) If V is an infinite-dimensional Hilbert space with orthonormal basis $\{f_i\}_{i=-\infty}^{\infty}$, show that the minimum mean square approximation to the vector f in V, using the subset $\{f_i\}_{i=-M}^{M}$ where $M < \infty$, is given by

$$f_M = \sum_{i=-M}^{M} c_i f_i \,,$$

where $\{c_i\}(-M \le i \le M)$ are coordinates of f i.e. $c_i = (f, f_i)$.

(*6 marks*)

(ii) The Hilbert space, V, of finite-power signals of period T has the inner product defined by

$$(f, g) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) g^*(t)\, \mathrm{d}t \,,$$

where f and g are finite power signals and * denotes complex conjugate. Assume that the set $\{f_n\}$ forms an orthonormal basis for V, where

$$f_n(t) = \mathrm{e}^{jnst} \qquad -\infty < n < \infty$$

and $s = 2p/T$.

The periodic function $f(t)$ in V is defined by

$$f(t) = t, \qquad\qquad |t| \le T/2 \,.$$

Find explicit expressions for the c_n which minimise the power of the error, $e(t)$, where

$$e(t) = f(t) - \sum_{n=-M}^{M} c_n f_n(t) \,.$$

Show that the relative minimum mean square error, $\|e\|^2 / \|f\|^2$, exceeds 10% unless $M \ge 6$.

(*19 marks*)

6.

(i) A random signal $f(t)$ is sampled at times $t = 0, T$ and $2T$ seconds, where $T = 1/2$, to produce the digital signal of length 3, $(f[0], f[1], f[2])$.
The autocorrelation function, $R_f(t)$, of $f(t)$ is given by

$$R_f(t) = \frac{s^2}{1 + |t| + 2t^2}.$$

Write down the correlation matrix, R, of $f[n]$, show that it is symmetric and that one of its eigenvalues is $\frac{3s^2}{4}$. Find the other eigenvalues.

(*13 marks*)

(ii) These digital signals are to be compressed using a single member of the Karhunen-Loeve basis. Find this basis vector and determine the minimum mean square error associated with this compression. (Work to 4 decimal places.)

(*12 marks*)

End of Question Paper