SCHOOL OF MATHEMATICS AND STATISTICS

Computational Engineering Mathematics

*Marks will be awarded for your best FOUR answers*
1 (i) The second order pde

\[ A \frac{\partial^2 \Phi}{\partial x^2} + B \frac{\partial^2 \Phi}{\partial x \partial y} + C \frac{\partial^2 \Phi}{\partial y^2} + D \frac{\partial \Phi}{\partial x} + E \frac{\partial \Phi}{\partial y} + F = 0, \]

where \( A, B, C, D, E \) and \( F \) are arbitrary constants, can be classified as being either elliptic, parabolic or hyperbolic according to the values of \( A, B \) and \( C \).

(a) For each of the three types of pde, give a simple example of a physical system which is modelled by that type. \( (3 \text{ marks}) \)

(b) State what conditions on \( A, B \) and \( C \) are required for the equation above to be elliptic, and state what additional conditions are then required to solve the problem. \( (3 \text{ marks}) \)

(ii) The one-dimensional diffusion equation, together with necessary additional conditions, is given by

\[ \frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \quad U(x, 0) = f(x), \quad U(0, t) = a, \quad U(1, t) = b \]

where \( \alpha \) is the diffusion coefficient. Using the standard notation that \( U_{ij} \equiv U(x_i, t_j) \) together with the conventions that \( i = 1 \) and \( i = n \) correspond to \( x = 0 \) and \( x = 1 \) respectively and that \( j = 1 \) corresponds to \( t = 0 \), use the standard finite difference approximations, given on the formulae sheet, together with the notation \( k = \Delta t / \Delta x^2 \), to derive the explicit scheme

\[ U_{ij} = \alpha k (U_{i+1,j-1} + U_{i-1,j-1}) + (1 - 2\alpha k) U_{ij-1}, \quad i = 2, \ldots, n-1, \quad j = 2, 3, \ldots \]

which approximates the differential equation. \( (3 \text{ marks}) \)

(iii) The diffusion equation is to be solved (approximately) over the range \( 0 \leq x \leq 1 \) for the temperature distribution along a given steel billet with boundary conditions \( U(0, t) = 20^\circ C \) and \( U(1, t) = 100^\circ C \) and initial conditions \( U(x, 0) = 80x^2 + 20 \), where it is assumed that the units have been normalized so that \( \alpha = 1 \). Assuming that we use \( \Delta x = 0.025 \) and \( \Delta t = 0.0005 \), then write a program which uses the explicit scheme to generate the approximate solution up to \( t = 1 \). You may use a programming language from amongst Scilab, Matlab, Fortran, Python or IDL. State clearly which language you are using. \( (11 \text{ marks}) \)
2 (i) A vanishingly small force, $\Delta f$, acts on a surface of vanishingly small area, $\Delta A$, drawn on the interior of a solid body. Using a diagram to clarify things, define what is meant by the stress at a point $P$ in $\Delta A$ and explain, briefly, why a complete mathematical description of stress requires it to be defined as a two-index tensor. 

(6 marks)

(ii) A concrete slab, of unit thickness in the $z$-direction, is loaded with body-forces $f$ and is in a state of plane stress so that $\sigma_{zz} = \sigma_{xz} = \sigma_{yz} = \sigma_{zx} = \sigma_{zy} = F_z = 0$. By considering only the balance of forces in the $x$-direction, use a diagram to derive the $x$-component of the equations of static equilibrium and hence infer for the general case (i.e. when $\sigma_{ij}, F_z \neq 0$) the full set of force-balance equations for a three-dimensional body.

(14 marks)
By reference to Figure 1, define the normal strain, $\varepsilon_{yy}$, and the engineering shear strain, $\gamma_{yx}$, and hence show that

$$\varepsilon_{yy} = \frac{\partial v}{\partial y}, \quad \gamma_{yx} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y},$$

where $u$ is the displacement in the $x$-direction and $v$ is the displacement in the $y$-direction of the body ABCD due to the stress forces acting on its surfaces. 

*(12 marks)*
(ii) The elastic constitutive matrix applying to the engineering strains for an isotropic material is given by

\[
C = \begin{bmatrix}
(\lambda + 2\mu) & \lambda & \lambda & 0 & 0 & 0 \\
(\lambda + 2\mu) & \lambda & 0 & 0 & 0 \\
(\lambda + 2\mu) & 0 & 0 & 0 \\
\mu & 0 & 0 \\
\mu & 0 \\
\mu & 0 
\end{bmatrix}
\]

where

\[
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \ \mu = \frac{E}{2(1+\nu)}.
\]

Given further that \(E = 33.0\, GPa\), \(\nu = 0.185\), and that a state of strain defined by \(\varepsilon_{xx} = 1010 \times 10^{-6}\), \(\varepsilon_{yy} = -0.28\varepsilon_{xx}\), \(\varepsilon_{zz} = -0.19\varepsilon_{xx}\), \(\varepsilon_{xy} = 227 \times 10^{-6}\), \(\varepsilon_{yz} = 427 \times 10^{-6}\) and \(\varepsilon_{zx} = -71 \times 10^{-6}\) exists at a point in a given isotropic material, calculate the corresponding state of stress at the point. \((8\, marks)\)
Figure 2: A rectangular plate with temperature defined on the boundaries.

Figure 2 shows a rectangular plate made of a homogeneous isotropic material. The temperature distribution in this plate satisfies the indicated boundary conditions (given in degrees centigrade) and has reached a steady-state condition so that it is described by Laplace’s equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0.$$

(i) Draw a sketch of the solution domain showing clearly the line of symmetry for the temperature distribution and indicating which of the unknown temperatures are equal to each other. (4 marks)

(ii) Use the finite difference formulae on the formulae sheet to formulate the finite difference equations required to find estimates of the nodal temperatures, $T_A, T_B, T_C$ and $T_D$. (10 marks)

(iii) Express these finite difference equations in the form $AT = B$ where $A$ is a $4 \times 4$ matrix, $T = (T_A, T_B, T_C, T_D)^T$ and $B = (-160, -160, -200, -240)^T$ is a $4 \times 1$ column vector. Find matrix $A$, hence, given that

$$A^{-1} \approx \begin{bmatrix}
-0.27 & -0.07 & -0.02 & -0.01 \\
-0.07 & -0.29 & -0.08 & -0.02 \\
-0.02 & -0.08 & -0.31 & -0.08 \\
-0.01 & -0.04 & -0.15 & -0.29
\end{bmatrix}$$

estimate $T_A, T_B, T_C$ and $T_D$ correct to one degree. (6 marks)
The velocity field in an unsteady moving fluid is given by \( \mathbf{V} = ui + vj + wk \), where 
\[ u \equiv u(x, y, z, t), \quad v \equiv v(x, y, z, t) \quad \text{and} \quad w \equiv w(x, y, z, t). \]

(i) By considering the density, \( \rho(x, y, z, t) \), and an infinitesimal control volume, 
\( \delta V \), in the fluid moving from a point \((x_1, y_1, z_1)\) at time \( t_1 \) to a point 
\((x_2, y_2, z_2)\) at time \( t_2 \), then derive the substantial (or total) derivative 
\[ \frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u \frac{\partial\rho}{\partial x} + v \frac{\partial\rho}{\partial y} + w \frac{\partial\rho}{\partial z} \]
of \( \rho \) and interpret the meanings of the first two terms on the right-hand side of this latter expression. \(12 \text{ marks}\)

(ii) \( \mathbf{G} \), the curl of a vector field \( \mathbf{F} \), may be expressed in index notation as 
\[ G_i = \varepsilon_{ijk} \frac{\partial F_k}{\partial x_j} \] where \( i, j, k \) may each take any of the values \( 1, 2, 3 \).

The components of the Levi-Civita tensor \( \varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1, \varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1 \), and are zero otherwise. A useful relation between the Levi-Civita tensor and the Kronecker delta is 
\[ \varepsilon_{ijk} \delta_{imn} = \delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km} \]
The vorticity of a fluid is \( \mathbf{\omega} = \nabla \times \mathbf{u} \), the curl of the velocity field \( \mathbf{u} \). The curl of the vorticity is \( \nabla \times \mathbf{\omega} \). Verify by index notation or otherwise that the curl of the vorticity may be expressed as \( \nabla(\nabla \cdot \mathbf{u}) - \nabla^2 \mathbf{u} \). \(8 \text{ marks}\)

End of Question Paper
Formulæ Sheet

Notation:

\[ U(x_i, t_j) \equiv U_{i,j} \]

Forward difference formula for \( \frac{\partial U}{\partial t} \):

\[
\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{i,j}}{\Delta t}
\]

Backward difference formula for \( \frac{\partial U}{\partial t} \):

\[
\frac{\partial U}{\partial t} \approx \frac{U_{i,j} - U_{i,j-1}}{\Delta t}
\]

Central difference formula for \( \frac{\partial U}{\partial x} \):

\[
\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}
\]

Central difference formula for \( \frac{\partial^2 U}{\partial x^2} \):

\[
\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{\Delta x^2}
\]