SCHOOL OF MATHEMATICS AND STATISTICS

Mathematics II (Electrical)

2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

_________ ___________ ___________ ___________
1 (i) The electrical charge in a circuit is described by

\[ q''(t) + 4q'(t) + 8q(t) = \delta(t) \]

subject to the initial conditions \( q(0) = q'(0) = 0 \).

(a) Show that the Laplace transform of \( q(t) \) is given by

\[ Q(s) = \frac{1}{2} \left( \frac{2}{(s + 2)^2 + 2^2} \right). \]

(7 marks)

(b) Use the inverse Laplace transform to determine the charge \( q(t) \) at time \( t > 0 \). (5 marks)

(ii) Let \( f(t) = e^{-|t|} \) and \( g(t) = \delta(t - 1) \).

(a) Use the convolution and time shift property of the Fourier transform to show that

\[ F\{f * g(t)\} = \frac{2e^{-j\omega}}{1 + \omega^2}. \]

(4 marks)

(b) Use the inverse Fourier transform to calculate \( f * g(t) \). Verify your result by the direct calculation of \( f * g(t) \). (4 marks)

2 (i) Let \( f : [0, \pi] \to \mathbb{R} \) be defined by \( f(t) = \pi - t \). Find the Fourier sine series of \( f(t) \). (15 marks)

(ii) Sketch the graph of the Fourier sine series of \( f(t) \) over the interval \([ -\pi, \pi ]\). Indicate where the Fourier sine series agrees with the odd extension of \( f(t) \) in the interval \([ -\pi, \pi ]\). (5 marks)

3 (i) Let \( f(x, y) = x^2 + y^2 + \sin (x^2) \). Find \( f_x(x, y) \) and \( f_y(x, y) \). (3 marks)

(ii) Find and classify all the critical points of the function

\[ f(x, y) = 2x^5 - y^5 - 10x + 5y \]

(10 marks)

(iii) Find \( \frac{\partial f}{\partial v} \) when \( f(x, y) = y(x + 2y) \), \( x(u, v) = \cos uv \) and \( y(u, v) = e^{u+v} \). (7 marks)
4 (i) Let $D \subset \mathbb{R}^2$ be the region bounded by the $x$-axis, $y$-axis, and the graph of the function $y = 1 - x^2$. Sketch the region $D$ and calculate its area. 

(7 marks)

(ii) Let $R$ be a cylinder bounded by the $xy$-plane, the plane $z = h$, and by the surface $x^2 + y^2 = R^2$. The mass density of the cylinder material is $\rho = \rho_0 e^{-\frac{(r/R)^2 + z/h}{2}}$. Calculate the cylinder mass $M$. 

(13 marks)

**Hint:** Use cylindrical coordinates $r, \theta, z$.

5 (i) Let $f(x, y) = x^2 - x^2 y - 2y^2$.

(a) Calculate the directional derivative of $f(x, y)$ at $(3, -1)$ in the direction of $v = (4, 3)$. 

(5 marks)

(b) In what direction is the graph of $f(x, y)$ most rapidly increasing at the point $(3, -1)$, and what is the maximum rate of increase? 

(5 marks)

(ii) Let $f(x, y, z) = xy$ and $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$\mathbf{F} = (xy, xz, yz)$.

Verify by the direct calculation that

$$\nabla \times (f \mathbf{F}) = f \nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F}.$$ 

(10 marks)

**End of Question Paper**
MAS241 FORMULA SHEET

Laplace transform:
The Laplace transform of a function $f(t)$ is given by:
\[
\mathcal{L}\{f(t)\}(s) := \int_{0}^{\infty} e^{-st} f(t) dt.
\]

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

| $\mathcal{L}\{af(t) + bg(t)\}$ | $a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$ | linearity |
| $\mathcal{L}\{f'(t)\}$ | $sF(s) - f(0)$ | differentiation w.r.t. $t$ |
| $\mathcal{L}\{f''(t)\}$ | $s^2F(s) - sf(0) - f'(0)$ | second differentiation w.r.t. $t$ |
| $\mathcal{L}\{e^{-kt}f(t)\}$ | $F(k + s)$ | frequency shift |
| $\mathcal{L}\{f(t-a)H(t-a)\}$ | $e^{-as}F(s)$ (for $a > 0$) | time shift |
| $\mathcal{L}\{f(at)\}$ | $\frac{1}{a}F\left(\frac{s}{a}\right)$ (for $a > 0$) | scaling |
| $\mathcal{L}\{f \ast g(t)\}$ | $\mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal) | convolution |

Table of standard Laplace transforms:

| $f(t)$ | $\mathcal{L}\{f(t)\}(s)$ | Region of validity |
| $t^n$ (for $n \geq 0$) | $\frac{n!}{s^{n+1}}$ | $Re(s) > 0$ |
| $\sin(kt)$ | $\frac{k}{s^2+k^2}$ | $Re(s) > 0$ |
| $\cos(kt)$ | $\frac{s}{s^2+k^2}$ | $Re(s) > 0$ |
| $H(t-T)$ (for $T \geq 0$) | $\frac{e^{-sT}}{s}$ | $Re(s) > 0$ |
| $\delta(t-T)$ (for $T \geq 0$) | $e^{-sT}$ | $s \in \mathbb{C}$ |

Fourier transform:
The Fourier transform and inverse Fourier transforms are given by:
\[
\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt, \quad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega.
\]

Convolution:
\[
f \ast g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau) d\tau.
\]
Properties of the Fourier transform: \( \mathcal{F}\{f(t)\} = F(\omega) \) in the following table:

| \( \mathcal{F}\{e^{j\theta t}f(t)\} \) | frequency shift |
| \( \mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega) \) | time shift |
| \( \mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega) \) | differentiation |
| \( \mathcal{F}\{F(t)\} = 2\pi f(-\omega) \) | symmetry |
| \( \mathcal{F}\{f(at)\} = \frac{1}{|a|} F(\frac{\omega}{a}) \) | scaling |
| \( \mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\} \) | convolution |

Table of standard Fourier transforms:

<table>
<thead>
<tr>
<th>( f(t) )</th>
<th>( \mathcal{F}{f(t)}(\omega) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{-a</td>
<td>t</td>
</tr>
<tr>
<td>rect(_T)(t)</td>
<td>( \sin(\frac{T\omega}{2}) )</td>
</tr>
<tr>
<td>1</td>
<td>( 2\pi \delta(\omega) )</td>
</tr>
<tr>
<td>( \delta(t) )</td>
<td>1</td>
</tr>
</tbody>
</table>

**Fourier series:**

The Fourier series of a periodic function \( f(t) \) with fundamental period \( T \) is given by

\[
S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \right)
\]

where

\[
\omega_n = \frac{2\pi n}{T}, \quad a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(\omega_n t) dt, \quad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(\omega_n t) dt.
\]

**Coordinate systems:**

**Cylindrical polar coordinates**

\[
(x, y, z) = (r \cos(\theta), r \sin(\theta), z)
\]

\[
(r, \theta, z) = \left( \sqrt{x^2 + y^2}, \arctan\left(\frac{y}{x}\right), z \right)
\]

\[
dV = r dr d\theta dz.
\]

**Spherical polar coordinates**

\[
(x, y, z) = (\rho \sin(\phi) \cos(\theta), \rho \sin(\phi) \sin(\theta), \rho \cos(\phi))
\]

\[
(\rho, \theta, \phi) = \left( \sqrt{x^2 + y^2 + z^2}, \arctan\left(\frac{y}{x}\right), \arccos\left(\frac{z}{\rho}\right) \right)
\]

\[
dV = \rho^2 \sin(\phi) d\rho d\phi d\theta.
\]