SCHOOL OF MATHEMATICS AND STATISTICS

MATHEMATICS III (CHEMICAL)

Attempt all the questions. The allocation of marks is shown in brackets.

1  (i) Find $\frac{dz}{dx}$, where $z = x^3y + \sin 2x$ and $y = \ln x$. 

(ii) Find $\frac{dy}{dx}$, given that

$$\cos^2 x + \cos^2 y = \cos(2x + 2y).$$

(iii) Write down the iteration formula for the Newton-Raphson method. Starting from $x_0 = 1.0$ use the Newton-Raphson method to find an approximation to a root of the equation

$$x^3 - x - 1 = 0,$$

correct to four decimal places.

2  (i) Find and classify the stationary points of the function

$$xy^2 - x^2 - 2y^2 = 0.$$

(ii) A periodic function, $f(x)$, of period $2\pi$ is defined by

$$f(x) = \pi^2 - x^2 \quad \text{for} \quad -\pi \leq x < \pi.$$

Supposing that $f(x)$ has a convergent trigonometric Fourier series, show that

$$\pi^2 - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx.$$
3 (i) The vector field, $\mathbf{F}$, is given by
$$\mathbf{F} = (2x + yz, xz, yx).$$

(a) Verify that $\nabla \times \mathbf{F} = 0$. \hspace{1cm} (3 marks)

(b) Find a scalar potential, $V$, such that $\mathbf{F} = \nabla V$. \hspace{1cm} (10 marks)

(ii) A vector field, $\mathbf{A}$, is given by
$$\mathbf{A} = (xy, 2xz^2, 3)$$
and a scalar field, $u$, is given by
$$u(x, y, z) = xyz.$$

(a) Evaluate $\nabla \cdot \mathbf{A}$ at the point with co-ordinates $(1, -3, 2)$. \hspace{1cm} (3 marks)

(b) Verify that
$$\nabla \times (u\mathbf{A}) = u(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla u.$$ \hspace{1cm} (9 marks)

4 Show that the partial differential equation
$$\frac{\partial^2 y}{\partial t^2} - 9 \frac{\partial^2 y}{\partial x^2} = 0,$$
has solutions of the form $y = f(x + \lambda t)$ for arbitrary functions $f$ provided that $\lambda = -3$ or $\lambda = 3$. \hspace{1cm} (6 marks)

Give an interpretation, including a clear diagram, of the form of the solution in each case. \hspace{1cm} (6 marks)

Derive the solution that satisfies the conditions
$$y(x, 0) = x^2,$$
$$\frac{\partial y}{\partial t} (x, 0) = \cos 3x.$$ \hspace{1cm} (13 marks)

End of Question Paper
Formula Sheet

Fourier Series
Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi x}{L} \, dx, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi x}{L} \, dx, \quad n = 1, 2, 3, \ldots.$$  

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad a_n = \frac{2}{L} \int_{0}^{L} f(x) \cos \frac{n\pi x}{L} \, dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad b_n = \frac{2}{L} \int_{0}^{L} f(x) \sin \frac{n\pi x}{L} \, dx$$

Gradient of a Scalar Field
The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \text{grad} \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$
Chain Rule

1. If \( z = f(x, y) \), where \( x = x(t), \ y = y(t) \), then

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}.
\]

2. If \( z = f(x, y) \), where \( x = x(u, v), \ y = y(u, v) \), then

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v}.
\]

3. If \( z = f(u, v) \), where \( u = u(x, y), \ v = v(x, y) \), then

\[
\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y}.
\]

Maxima and Minima

1. The function \( f(x, y) \) has a stationary point at \((x_0, y_0)\) if

\[ f_x = f_y = 0 \quad \text{at} \ (x_0, y_0). \]

2. At \((x_0, y_0)\), the function \( f(x, y) \) has:

(i) a minimum if

\[ f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \ (x_0, y_0), \]

(ii) a maximum if

\[ f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \ (x_0, y_0), \]

(iii) a saddle point if

\[ f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \ (x_0, y_0). \]