SCHOOL OF MATHEMATICS AND STATISTICS  
Autumn Semester  
2013–14  
Mathematics II (Materials)  
2 hours  

Marks will be awarded for answers to all questions in Section A, and for your best THREE answers to questions in Section B. Section A carries 40 marks, and the marks awarded to each question or section of question are shown in italics.

Section A

A1 Find the solution of the equation

\[ \frac{dy}{dx} + xy = e^{-\frac{1}{2}x^2} \]

for \( x > 0 \) which satisfies \( y = 1 \) when \( x = 1 \).  

(7 marks)

A2 Find the general solution of the equation

\[ 9 \frac{d^2y}{dx^2} - 12 \frac{dy}{dx} + 4y = 0. \]

(5 marks)

A3 If

\[ f(x, y) = xy^2 - 2 \sin(x + 3y) \]

and

\[ x = \frac{s}{r}, \quad y = rs, \]

use the chain rule to find \( \frac{\partial f}{\partial r} \) and \( \frac{\partial f}{\partial s} \), giving your answers in terms of \( r \) and \( s \), and simplifying where possible.  

(10 marks)

Find also \( \frac{\partial^2 f}{\partial r^2} \) in terms of \( r \) and \( s \).  

(3 marks)
The following table shows the wage bills (in millions of pounds) and the final points totals of 10 Premier League football clubs for 2007-08:

<table>
<thead>
<tr>
<th>Club</th>
<th>Wages (x)</th>
<th>Points (y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Arsenal</td>
<td>101.3</td>
<td>83</td>
</tr>
<tr>
<td>Aston Villa</td>
<td>50.4</td>
<td>60</td>
</tr>
<tr>
<td>Blackburn Rovers</td>
<td>39.7</td>
<td>58</td>
</tr>
<tr>
<td>Bolton Wanderers</td>
<td>39</td>
<td>37</td>
</tr>
<tr>
<td>Chelsea</td>
<td>149</td>
<td>85</td>
</tr>
<tr>
<td>Everton</td>
<td>44.5</td>
<td>65</td>
</tr>
<tr>
<td>Fulham</td>
<td>39.3</td>
<td>36</td>
</tr>
<tr>
<td>Liverpool</td>
<td>80</td>
<td>76</td>
</tr>
<tr>
<td>Manchester City</td>
<td>54.2</td>
<td>55</td>
</tr>
<tr>
<td>Manchester United</td>
<td>121.1</td>
<td>87</td>
</tr>
</tbody>
</table>

Calculate the means and variances of $x$ and $y$, and also the covariance between $x$ and $y$. \( (10 \text{ marks}) \)

It is assumed that $x$ and $y$ satisfy the linear relationship

\[ y = a + b(x - \bar{x}), \quad (*) \]

where $\bar{x}$ is the mean of $x$.

Calculate the least squares estimates of $a$ and $b$, correct to 3 significant figures. State, giving reasons, whether you expect (*) to give a good model. \( (5 \text{ marks}) \)

**Section B**

**B1**

(a) Find the general solution of the equation

\[ y \frac{dy}{dx} + y^2 = 1 \quad \text{for } |y| > 1. \quad (8 \text{ marks}) \]

(b) Find the solution of the equation

\[ \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} - 4y = e^{-4x}, \]

given that $y = 0$ and $\frac{dy}{dx} = 1$ when $x = 0$. \( (12 \text{ marks}) \)
B2 (a) Find a vector normal to the surface \( \phi = 2 \) at the point \( A \) with coordinates \((2, 0, -1)\), where
\[
\phi = x \sin y + y^2 + xz^2.
\]
Hence find the equation of the tangent plane to the surface at \( A \). 
(5 marks)

Find also the directional derivative of \( \phi \) at \( A \), in the direction \( \mathbf{d} = (1, 2, 2) \). 
(2 marks)

(b) A scalar field \( \psi \) and a vector field \( \mathbf{u} \) are given by
\[
\psi = x \sinh y + y \cos z + x \sin z,
\quad \mathbf{u} = (x^2 y, y^2 z, z^2 x).
\]
Verify that
\[
\nabla \times \nabla \psi = 0
\]
and
\[
\nabla \cdot (\nabla \times \mathbf{u}) = 0.
\]
(11 marks)

B3 A function \( f(x) = x^2 \) is defined on the interval \(-1 \leq x \leq 1\).

(a) Show that \( f(x) \) can be represented by the Fourier series
\[
\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x).
\]
[You may use the fact that \( \cos(n\pi) = (-1)^n \).] 
(18 marks)

(b) Sketch the function given by the above Fourier series on the interval 
\(-3 \leq x \leq 3\). 
(2 marks)

B4 A string of unstretched length \( l > 0 \) has its ends fixed at \( x = 0 \) and \( x = l \). Its displacement \( y(x, t) \) in the transverse direction satisfies the equation
\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]
where \( c \) is a constant.

(a) Using separation of variables, show that the general solution can be written as
\[
y(x, t) = \sum_{n=1}^{\infty} \sin \left( \frac{n\pi x}{l} \right) \left[ D_n \cos \left( \frac{n\pi ct}{l} \right) + E_n \sin \left( \frac{n\pi ct}{l} \right) \right]
\]
where \( D_n \) and \( E_n \) are constants. 
(14 marks)

(b) If \( y = 0 \) and \( \frac{\partial y}{\partial t} = 2c \sin \left( \frac{2\pi x}{l} \right) \) at \( t = 0 \) show that
\[
y(x, t) = \frac{l}{\pi} \sin \left( \frac{2\pi x}{l} \right) \sin \left( \frac{2\pi ct}{l} \right). \]
(6 marks)

End of Question Paper
FORMULA SHEET

Trigonometry
\[ 1 + \tan^2 \theta = \sec^2 \theta \]
\[ 1 + \cot^2 \theta = \cosec^2 \theta \]
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]
\[ \cos(A - B) = \cos A \cos B + \sin A \sin B \]
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]
\[ \sin(A - B) = \sin A \cos B - \cos A \sin B \]
\[ \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \]
\[ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \]
\[ \sin 2\theta = 2 \sin \theta \cos \theta \]
\[ \cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \]
\[ a \cos \theta + b \sin \theta = R \cos(\theta - \alpha), \text{ where } R = \sqrt{(a^2 + b^2)}, \cos \alpha = a/R \text{ and } \sin \alpha = b/R \]

Hyperbolic Functions
\[ \sinh x = \frac{1}{2}(e^x - e^{-x}) \]
\[ \cosh x = \frac{1}{2}(e^x + e^{-x}) \]
\[ \cosh^2 x - \sinh^2 x = 1 \]
\[ \text{sech}^2 x + \tanh^2 x = 1 \]
\[ 2 \sinh x \cosh x = \sinh 2x \]
\[ \cosh 2x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1 \]
\[ \sinh^{-1} x = \ln \left[ x + \sqrt{(1 + x^2)} \right], \text{ all } x \]
\[ \cosh^{-1} x = \ln \left[ x + \sqrt{(x^2 - 1)} \right], \quad x \geq 1 \]
\[ \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \quad |x| < 1 \]
\[ \coth^{-1} x = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right), \quad |x| > 1 \]
## Differentiation and Integration

<table>
<thead>
<tr>
<th>Function</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$\ln x$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$\tan x$</td>
<td>$\sec^2 x$</td>
</tr>
<tr>
<td>$\cot x$</td>
<td>$-\csc^2 x$</td>
</tr>
<tr>
<td>$\sec x$</td>
<td>$\sec x \tan x$</td>
</tr>
<tr>
<td>$\csc x$</td>
<td>$-\csc x \cot x$</td>
</tr>
<tr>
<td>$\sinh x$</td>
<td>$\cosh x$</td>
</tr>
<tr>
<td>$\cosh x$</td>
<td>$\sinh x$</td>
</tr>
<tr>
<td>$\tanh x$</td>
<td>$\sech^2 x$</td>
</tr>
<tr>
<td>$\coth x$</td>
<td>$-\cosech^2 x$</td>
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<td>$\sech x$</td>
<td>$-\sech x \tanh x$</td>
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<tr>
<td>$\cosech x$</td>
<td>$-\cosech x \coth x$</td>
</tr>
<tr>
<td>$\sin^{-1} x$</td>
<td>$\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\cos^{-1} x$</td>
<td>$-\frac{1}{\sqrt{1-x^2}}$</td>
</tr>
<tr>
<td>$\tan^{-1} x$</td>
<td>$\frac{1}{1+x^2}$</td>
</tr>
<tr>
<td>$\cot^{-1} x$</td>
<td>$-\frac{1}{1+x^2}$</td>
</tr>
<tr>
<td>$\sinh^{-1} x$</td>
<td>$\frac{1}{\sqrt{x^2+1}}$</td>
</tr>
<tr>
<td>$\cosh^{-1} x$</td>
<td>$\frac{1}{\sqrt{x^2-1}}$</td>
</tr>
<tr>
<td>$\tanh^{-1} x$</td>
<td>$\frac{1}{1-x^2}$, $</td>
</tr>
<tr>
<td>$\coth^{-1} x$</td>
<td>$-\frac{1}{x^2-1}$, $</td>
</tr>
</tbody>
</table>
Function | Integral
--- | ---
$\frac{1}{a^2 + x^2}$ | $\frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right)$
$\frac{1}{a^2 - x^2}$ | $\frac{1}{a} \tanh^{-1} \left( \frac{x}{a} \right)$
$\frac{1}{\sqrt{a^2 - x^2}}$ | $\sin^{-1} \left( \frac{x}{a} \right)$
$\frac{1}{\sqrt{a^2 + x^2}}$ | $\sinh^{-1} \left( \frac{x}{a} \right)$
$\frac{1}{\sqrt{x^2 - a^2}}$ | $\cosh^{-1} \left( \frac{x}{a} \right)$

Differentiation and Integration Formulae

\[
\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}
\]

\[
\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{\frac{du}{dx} - \frac{uv}{dx}}{v^2}
\]

\[
\int_a^b uv \, dx = \left[ u \times (\text{integral of } v) \right]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v) \, dx
\]

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that $x$ and $y$ are functions of $t$, i.e., $x = x(t)$, $y = y(t)$. Then

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

2. Suppose that $z = f(x, y)$ and that $x$ and $y$ are functions of the variables $r$ and $s$, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}, \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]
First-Order Differential Equations

1. Direct Integration

\[
\frac{dy}{dx} = f(x)
\]

\[
y = \int f(x) \, dx + C
\]

2. Separation of Variables

\[
\frac{dy}{dx} = f(x)g(y)
\]

\[
\int \frac{dy}{g(y)} = \int f(x) \, dx
\]

3. Homogeneous Equations

\[
\frac{dy}{dx} = f \left( \frac{y}{x} \right)
\]

make the substitution \( y = zx \) to give

\[
z + x \frac{dz}{dx} = f(z)
\]

4. Linear Equations

\[
\frac{dy}{dx} + P(x)y = Q(x)
\]

multiply both sides by the integrating factor \( e^{\int P(x) \, dx} \) to give

\[
\frac{d}{dx} \left( ye^{\int P(x) \, dx} \right) = Q(x)e^{\int P(x) \, dx}
\]
The Second-Order Differential Equation

\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \]

where \( a, b, \) and \( c \) are constants.

General solution is

\[ y = \text{Complementary Function} + \text{Particular Integral} \]

The solution, \( y_c \), is given by

(i) \( y_c = Ae^{m_1x} + Be^{m_2x} \), if \( m_1 \) and \( m_2 \) real and different,

(ii) \( y_c = e^{mx}(A + Bx) \), if \( m_1 \) and \( m_2 \) real and equal \( (m_1 = m_2 = m) \),

(iii) \( y_c = e^{px}(A \cos mx + B \sin mx) \), if \( m_1 \) and \( m_2 \) are complex \( (m_1 = p + iq, m_2 = p - iq) \),

where \( m_1 \) and \( m_2 \) are the roots of the auxiliary equation

\[ am^2 + bm + c = 0 \]

Particular Integral, \( y_p \)

\[ f(x) = Ax^2 + Bx + C \quad y_p = ax^2 + bx + c \]

\[ f(x) = Ae^{kx} \quad y_p = ae^{kx} \]

when \( k \) is not one of the roots of the auxiliary equation

\[ f(x) = Ae^{kx} \quad y_p = ake^{kx} \]

when \( k \) is one of the roots of the auxiliary equation

\[ f(x) = A \cos mx + B \sin mx \quad y_p = a \cos mx + b \sin mx \]

when \( \sin mx \) or \( \cos mx \) is not part of the complementary function

\[ f(x) = A \cos mx + B \sin mx \quad y_p = x(a \cos mx + b \sin mx) \]

when \( \sin mx \) or \( \cos mx \) is part of the complementary function
**Fourier Series**

Suppose that \( f(x) \) is defined on the interval \(-l \leq x \leq l\). The Fourier series for \( f(x) \) is given by

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),
\]

where

\[
a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad n = 0, 1, 2, \ldots,
\]

\[
b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx, \quad n = 0, 1, 2, \ldots.
\]

On the interval \( 0 \leq x \leq l \) the Fourier cosine series for \( f(x) \) is

\[
f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} \, dx
\]

and the Fourier sine series is

\[
f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx.
\]

**Vector Calculus**

The gradient of the scalar field \( \phi(x, y, z) \) is given by

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).
\]

The divergence of a vector field \( \mathbf{u}(x, y, z) = (u, v, w) \) is given by

\[
\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}
\]

The curl of a vector field \( \mathbf{u}(x, y, z) = (u, v, w) \) is given by

\[
\nabla \times \mathbf{u} = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
\partial x & \partial y & \partial z \\
u & v & w
\end{vmatrix}
\]

The Laplacian \( \nabla^2 \) is given by

\[
\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}
\]
Statistics

For data values \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

Means \(\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i\) etc.

Variances \(s_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i^2) - \bar{x}^2\) etc.

\(s_x\) is standard deviation

Covariance \(\text{cov}(x, y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n} \sum_{i=1}^{n} (x_iy_i) - \bar{x}\bar{y}\)

Correlation coefficient \(r = \frac{\text{cov}(x, y)}{s_x s_y}\)

Linear regression by least squares

The least squares fit to the linear relationship

\[ y = a + b(x - \bar{x})\]

is given by

\[ a = \bar{y}, \quad b = \frac{\text{cov}(x, y)}{s_x^2}\]

The corresponding mean square residual is \(s_y^2(1 - r^2)\).