1 (i) If \( f(x, y, z) = 2\frac{y}{x-z} - \sin \frac{xz}{y} \) and 
\[ x = \frac{3t}{t^2 + 1}, y = e^{2t-2}, z = \ln t, \quad (t > 0), \]
use the chain rule to evaluate \( \frac{df}{dt} \) when \( t = 1 \). \( \text{(14 marks)} \)

(ii) The power of an engine is given in terms of three parameters as 
\[ w(x, y, z) = \frac{x - 2z}{\sqrt{1 + y^2}} - \ln \frac{x}{y}. \]

During the functioning of the engine the value of the parameter \( x \) increases from 1 to 1.2, the value of the parameter \( y \) decreases from 0.6 to 0.5 while the initial value of \( z \) is 0.3. Use the small error formula to find the variation of the third parameter, \( z \), so that the power output of the engine does not change. Work with an accuracy of 2 decimal places. \( \text{(11 marks)} \)
A function of $x$ and $y$ which satisfies Laplace’s equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

is given in the form

$$(A \cos sx + B \sin sx)(C \cosh sy + D \sinh sy)$$

(1)

where $A$, $B$, $C$ and $D$ are arbitrary constants and $s$ is a parameter. If Laplace’s equation is satisfied in the rectangular region $0 \leq x \leq a$, $0 \leq y \leq b$ and the boundary conditions on $x = 0$ and $x = a$ are given by $\Phi = 0$, use expression (1) to show that the the solution, $\Phi(x, y)$, of Laplace’s equation which satisfies these conditions is

$$\Phi(x, y) = \sum_{n=1}^{\infty} \left( c_n \cosh \frac{n\pi y}{a} + d_n \sinh \frac{n\pi y}{a} \right) \sin \frac{n\pi x}{a}$$

where $c_n$, $d_n$ are constants.

Given that on $y = 0$ the boundary condition is $\Phi = 1$, determine the constants $c_n$. (12 marks)

With the value of $c_n$ determined before, find the constants $d_n$ provided $\Phi = x$ on $y = b$. (13 marks)

(i) Find the Fourier series decomposition of the function $f(x) = x^2 - x$ in the interval $-\pi \leq x \leq \pi$. (15 marks)

(ii) With the series decomposition obtained at part (i) prove that in the case of $x = 0$ the equality

$$\pi^2 = 12 \left( 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \ldots \right)$$

is valid. (4 marks)

(iii) Values of $y$ at $x = 2$ determined using the fourth-order Runge-Kutta method with two different step-lengths are given in the following table

<table>
<thead>
<tr>
<th>$h$</th>
<th>$y(2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.2</td>
<td>3.40978</td>
</tr>
<tr>
<td>0.4</td>
<td>3.39278</td>
</tr>
</tbody>
</table>

Use this data to estimate a value for $h$ which will ensure that the error in the calculated value of $y(2)$ using a fourth-order Runge-Kutta method does not exceed $10^{-4}$. Give your answer correct to 4 decimal places. (6 marks)
(i) The function $y(x)$ satisfies the differential equation

\[ \frac{dy}{dx} = x + \sin y + 0.6, \]

and the condition $y = 1.2$ when $x = 0$. Obtain the first four terms of the Taylor series solution of this equation, working correct to four decimal places. **NOTE**: Remember to put your calculator in radian mode when calculating $\sin y$. 

(10 marks)

(ii) An explicit approximation to the heat conduction equation

\[ \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad 0 < x < 1 \]

is given (in the usual notations) by

\[ u_{i,j+1} = ru_{i-1,j} + (1 - 2r)u_{i,j} + ru_{i+1,j}, \quad u_{i,j} = u(x_i,t_j) \]

where $r = k/h^2$. If the initial and boundary conditions associated with the heat conduction equation are given by

\[ u(x,0) = 2x^2, \quad 0 < x < 1 \]
\[ u(0,t) = 0, \quad \text{and} \quad u(1,t) = 2, \quad t > 0 \]

use the above explicit scheme, with $h = 0.2$ and $k = 0.008$ to calculate grid-points values of $u$ at the first time step. Work correct to 3 decimal places. 

(7 marks)

(iii) Show that the equation

\[ x^2 - e^x - 6 = 0 \]

has a root in the interval $(-2.6, -2.2)$ and perform five iterations of the bisection method to obtain a refined estimate of the interval which contains the root. Work correct to three decimal places. 

(8 marks)

End of Question Paper
Formula sheet

- The local truncation error in the case of the 4th order Runge-Kutta method is given by
  \[ Y(x) - y(x) = Ch^4 \]
  where \( Y(x) \) is the exact value, \( y(x) \) is the estimated numerical value, \( C \) is a constant and \( h \) is the step size used in the numerical scheme.

- Chain rule
  If \( z = f(x, y) \), where \( x \) and \( y \) are both functions of \( t \), so that \( x = x(t) \) and \( y = y(t) \) we have
  \[ \frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt} \]
  If \( z = f(x, y) \) and both \( x \) and \( y \) are functions of \( u \) and \( v \), so that \( x = x(u, v) \) and \( y = y(u, v) \) then we have
  \[ \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} \]
  \[ \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial v} \]

- Fourier series
  If the function \( f(x) \) is defined over the interval \(-l \leq x \leq l\), then the Fourier series of \( f(x) \) is
  \[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}) \]
  where
  \[ a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad (n = 0, 1, 2, \ldots) \]
  \[ b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \quad (n = 1, 2, 3, \ldots) \]
  If the function \( f(x) \) is defined over the interval \( 0 \leq x \leq l \), then the Fourier cosine series of \( f(x) \) is
  \[ f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad (n = 0, 1, 2, \ldots) \]
  while the sine series of \( f(x) \) is
  \[ f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx \quad (n = 1, 2, 3, \ldots) \]