Answer four questions. If you answer more than four questions, only your best four will be counted.
1. (i) Express both of the following in the form \( x + iy \):

\[
\frac{13 - i}{1 - 2i}; \quad (1 - i)^{11}.
\]


(ii) Express

\[
\frac{(1 - i)^{13}}{(\sqrt{3} - i)^{11}}
\]

in the form \( re^{i\theta} \) with \( r > 0 \) and \(-\pi < \theta \leq \pi\).


(iii) State, without proof, the triangle inequalities for \( |z + w| \) and \( |z - w| \).

Show that, if \( |z| \leq 1 \), then

\[
\frac{1}{5} \leq \left| \frac{3z - 4}{2z + 3} \right| \leq 7.
\]

(iv) Write down the definitions of \( \cosh z \) and \( \sinh z \).

Find all the solutions of the following equation:

\[
2 \cosh z + \sinh z = i.
\]

(v) The path \( \gamma \) is the arc of the circle \( |z + 1| = 1 \) from 0 to -2 given by \( z = -1 + e^{it} \) \((0 \leq t \leq \pi)\). Evaluate

\[
\int_{\gamma} zdz, \quad \int_{\gamma} z^3 \cos(z^4)dz.
\]

(vi) Find all the sixth roots of \(-1\). Hence express \( x^6 + 1 \) as the product of three real quadratic factors.
2 (i) State, without proof, the Cauchy-Riemann equations for a differentiable function. (1 mark)

(a) Let \( g(z) = 4z - 3\bar{z} \) for all \( z \in \mathbb{C} \). Prove that \( g \) is nowhere differentiable. (3 marks)

(b) The function \( h \) is analytic in the complex plane and
\[
\text{Im}(h(z)) + \text{Re}(h(z)) = 2 \quad \text{for all} \quad z \in \mathbb{C}.
\]
Show that \( h \) is constant. (5 marks)

(ii) In each of the following cases, determine whether there is a function \( k \) analytic on \( \mathbb{C} \) with \( \text{Re}(k(x + iy)) = u(x, y) \), giving reasons for your answers:

(a) \( u(x, y) = \cosh x \cosh y \),
(b) \( u(x, y) = x^3 - 3xy^2 - 2y + 1 \).

When \( k \) exists, find an explicit expression for \( k(z) \) in terms of \( z \) and show that you have found all functions the satisfying the conditions. (8 marks)

(iii) Let the path \( \alpha \) from 1 to \( -3 \), consist of the straight line segment from 1 to \( 1 + 3i \), followed by the straight line segment from \( 1 + 3i \) to \( -3 + 3i \), followed by the straight line segment from \( -3 + 3i \) to \( -3 \). Sketch \( \alpha \). Use the ML estimate to show that
\[
\left| \int_{\alpha} \frac{e^z \sin z}{z^2} \, dz \right| \leq 10 e \cosh 3.
\] (8 marks)
State, without proof, Cauchy’s Theorem and Cauchy’s Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. (7 marks)

Let \( \gamma \) be the square contour with vertices 2, 2i, -2, -2i described in the anticlockwise direction. Without using the Residue Theorem, evaluate

(i) \( \int_\gamma \frac{\sin(\pi z)}{3z - 1} \, dz \),    (ii) \( \int_\gamma \frac{e^z + 1}{z^2 + 9} \, dz \),

(iii) \( \int_\gamma \frac{e^z}{z^2(z + 3)} \, dz \),    (iv) \( \int_\gamma \frac{e^z}{z(z + 1)} \, dz \).

(14 marks)

Let the contour \( \alpha \) be the circle \(|z - 1| = 2\) described in the positive direction. Evaluate

\[ \int_\alpha (z^2 + \bar{z}) \, dz. \]

(4 marks)
4. (i) Let \( f \) have a pole of order \( k \) at \( \alpha \). Prove that the residue of \( f \) at the point \( \alpha \) is given by

\[
\text{Res}\{f; \alpha\} = \frac{1}{(k-1)!} \lim_{z \to \alpha} \frac{d^{k-1}}{dz^{k-1}}[(z - \alpha)^k f(z)].
\]

(5 marks)

(ii) For each of the following functions, find all the singularities in \( \mathbb{C} \). Classify these singularities giving reasons for your answers and evaluate the residue at each of them:

(a) \( \frac{\cos(\pi z)}{e^z (z - 1)^2} \),

(4 marks)

(b) \( z \exp\left(\frac{1}{z - 1}\right) \),

(4 marks)

(c) \( \frac{e^{\pi z}}{e^{\pi z} + 1} \),

(5 marks)

(d) \( \frac{1 + \cos(\pi z)}{(z - 1)^2} \),

(3 marks)

(e) \( \frac{1 + \cos(\pi z)}{(z - 1)^5} \).

(4 marks)
5 (i) State, without proof, Cauchy’s Residue Theorem. Your statement should include conditions under which the result is valid. (4 marks)

Let \( \gamma \) be the triangular contour with \textbf{vertices} 2, 2i, -2i described in the anti-clockwise direction. Evaluate

\[
\int_{\gamma} \frac{\sin \pi z}{(2z + 1) \cos \pi z} \, dz , \quad \int_{\gamma} (z + 1) \cos \left( \frac{1}{z - 1} \right) \, dz .
\]

(11 marks)

(ii) Prove that

\[
\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + 1)(x^2 + 4)} \, dx = \frac{\pi(e - 1)}{3e^2} .
\]

(10 marks)

\textbf{End of Question Paper}