Answer four questions. If you answer more than four questions, only your best four will be counted.

A list of formulae is provided on the last page.

1. (i) The curve in the following picture is parametrised by \((a \cos^3 t, a \sin^3 t)\) for some suitable values of \(t\). Find its total arc length. \((7 \text{ marks})\)

(ii) Find the maximum and minimum values of curvature on the ellipse defined by \(x^2 + 4y^2 = 1\). \((9 \text{ marks})\)

(iii) A curve on \(\mathbb{R}^2\) is said to be self-similar if it is congruent to its own image under any map of the form

\[ \varphi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \varphi(x, y) = (\lambda x, \lambda y), \quad \text{where } \lambda > 0. \]

Let \(C\) be a curve on \(\mathbb{R}^2\) and \(k(s)\) be its curvature function in terms of a unit-speed parameter \(s > 0\). Prove that, if \(k(\lambda s) = \lambda^{-1}k(s)\) for \(s, \lambda > 0\), then \(C\) is self-similar. \((9 \text{ marks})\)
Consider the following vector-valued function

\[ \varphi(u, v) = \left( u, \sqrt{v^2 + 1}, \ln(v + \sqrt{v^2 + 1}) \right). \]

(a) Show that \( \varphi \) defines a local isometry between \( \mathbb{R}^2 \) and some surface \( S \subset \mathbb{R}^3 \). (5 marks)

(b) Find all the geodesics on \( S \) that pass through \((0, 1, 0)\). (7 marks)

(c) What is the arc length of the shortest path on \( S \) between \((0, 1, 0)\) and \((1, \sqrt{2}, \ln(1 + \sqrt{2}))\)? (4 marks)

(ii) Given a surface in \( \mathbb{R}^3 \), a curve on the surface is called a normal section if

- the curve is the intersection of the surface with a plane, and
- the normal vectors of the surface along the curve are parallel to the same plane.

Prove that any normal section is a pre-geodesic. (9 marks)

Let \( S \) be the surface (catenoid) parametrised by

\[ x(t, \theta) = (\cosh t \cos \theta, \cosh t \sin \theta, t). \]

(a) Show that the first fundamental form of this parametrisation is \( \cosh^2 t \left( dt^2 + d\theta^2 \right) \). (2 marks)

(b) Find the area of the region on \( S \) between the latitudes \( t = 0 \) and \( t = \ln 2 \). (7 marks)

(c) Let \( S' \) be the cylinder in \( \mathbb{R}^3 \) defined by \( x^2 + y^2 = 1 \). Find a function \( f \) such that the following map

\[ \varphi : S \to S', \quad \varphi(x(t, \theta)) = (\cos \theta, \sin \theta, f(t)) \]

preserves the areas of all regions. (8 marks)

(ii) Let \( X(u, v) \) and \( Y(u, v) \) be two functions that satisfy

\[ X_u(u, v) = Y_v(u, v), \quad X_v(u, v) = -Y_u(u, v). \]

Show that the map \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) defined by \( \psi(u, v) = (X(u, v), Y(u, v)) \) is conformal. (8 marks)
4 (i) Suppose a surface in $\mathbb{R}^3$ has a parametrisation $x(u,v)$ whose first and second fundamental forms are respectively

$$(u^2 + 1)^2(du^2 + dv^2), \quad 2du^2 + (u^2 - 1)dv^2.$$ 

(a) Find the principal curvatures as functions of $(u,v)$. \hspace{1cm} (6 marks)

(b) For what values of $(u,v)$ are all tangential directions principal? \hspace{1cm} (3 marks)

(c) For what values of $(u,v)$ are there tangential directions with zero normal curvature? \hspace{1cm} (6 marks)

(ii) Prove that, if a surface in $\mathbb{R}^3$ has a parametrisation whose second fundamental form is zero everywhere, then it must be contained in a plane. \hspace{1cm} (10 marks)

5 (i) The standard unit sphere $S^2$ can be parametrised by

$$x(\phi, \theta) = (\cos \phi \cos \theta, \cos \phi \sin \theta, \sin \phi)$$

whose first and second fundamental forms are both $d\phi^2 + \cos^2 \phi d\theta^2$.

(a) Find the Gaussian curvature of $S^2$ as a function of $(\phi, \theta)$. \hspace{1cm} (5 marks)

(b) Does there exist any local isometry between some part of $S^2$ and some part of a plane? Briefly explain why. \hspace{1cm} (4 marks)

(c) Find all the latitudes on $S^2$ (i.e., curves of the form $\phi = \text{constant}$) along which parallel transport takes any tangent vector $v$ to $-v$. \hspace{1cm} (7 marks)

(ii) Let $S$ be a surface whose Gaussian curvature is $-1$ everywhere. Prove that the area of any geodesic triangle on $S$ is less than $\pi$. \hspace{1cm} (9 marks)

End of Question Paper
List of Formulae

For a curve on \( \mathbb{R}^2 \) parametrised by \( \mathbf{x}(t) = (x(t), y(t)) \):

- arc length from \( \mathbf{x}(a) \) to \( \mathbf{x}(b) \)
  \[
  \int_a^b \| \mathbf{x}'(t) \| \, dt
  \]

- curvature
  \[
  k(t) = \frac{x'(t)y''(t) - y'(t)x''(t)}{[x'(t)^2 + y'(t)^2]^{3/2}}
  \]

For a surface in \( \mathbb{R}^3 \) parametrised by \( \mathbf{x}(u, v) \):

- first fundamental form
  \[
  E du^2 + 2F du dv + G dv^2, \quad E = \mathbf{x}_u \cdot \mathbf{x}_u, \quad F = \mathbf{x}_u \cdot \mathbf{x}_v, \quad G = \mathbf{x}_v \cdot \mathbf{x}_v
  \]

- surface areas
  \[
  \iint \sqrt{EG - F^2} \, du dv
  \]

- second fundamental form
  \[
  L du^2 + 2M du dv + N dv^2, \quad L = \mathbf{x}_{uu} \cdot \mathbf{n}, \quad M = \mathbf{x}_{uv} \cdot \mathbf{n}, \quad N = \mathbf{x}_{vv} \cdot \mathbf{n}
  \]
  where \( \mathbf{n} = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\| \mathbf{x}_u \times \mathbf{x}_v \|} \)

- Weingarten matrix
  \[
  W = \begin{bmatrix} E & F \\ F & G \end{bmatrix}^{-1} \begin{bmatrix} L & M \\ M & N \end{bmatrix}
  \]

- Gaussian curvature
  \[
  K = \det W
  \]

The Gauss-Bonnet formula for a compact region \( R \) on a surface:
\[
\iint_R K \, dA + \int_{\partial R} k_g ds + \sum \text{turning angles} = 2\pi \chi(R)
\]