Financial Mathematics

2 hours and 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

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Registration number from U-Card (9 digits)
to be completed by student

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1. Consider a perpetual bond that pays £5 once a year, every year, and whose first payment occurs in 6 months. Assume that spot interest rates for all maturities are 3%.

(i) Find the price of the bond. \( \text{(8 marks)} \)

(ii) Consider a \( N \)-year forward contract on the perpetual bond, where \( N \) is a positive integer. Show that the correct forward price in this forward contract is identical to the spot price of the bond. \( \text{(8 marks)} \)

(iii) You are given the opportunity to take a long position in a two-year forward contract as in (ii) at a forward price of £160. Describe in detail an arbitrage opportunity available to you. \( \text{(9 marks)} \)

2. (i) (a) Describe a portfolio consisting entirely of European put options on the same stock, with same expiration time \( T > 0 \), but with different strike prices, and whose payoff at time \( T \) as a function of \( S \), the spot price of the stock at time \( T \), is described by the graph below. \( \text{(4 marks)} \)

![Payoff Graph](image)

(b) Let \( p_{20}, p_{30}, p_{40} \) and \( p_{60} \) be the prices of the above put options with strike prices 20, 30, 40 and 60, respectively, and let \( c_{10} \) be the price of a European call option on the same stock, with expiration at time \( T \) and with strike price 10. By comparing the payoff of the portfolio in (a) and the payoff of the call option above, describe an inequality involving \( c_{10}, p_{20}, p_{30}, p_{40} \) and \( p_{60} \). \( \text{(5 marks)} \)

(ii) Describe what American put options are. \( \text{(3 marks)} \)

(iii) The price of a stock which pays no dividends is currently £20. Over each of the next three 1-year periods the stock price will either increase by 10% or decrease by 10%. Suppose that all interest rates are constant and equal to 3%.

(a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £20. \( \text{(11 marks)} \)

(b) Describe all circumstances when a rational investor should exercise the option described in (a) before its expiration. \( \text{(2 marks)} \)
3 (i) Consider a derivative on a stock which entitles the holder to one payoff at time \( T \); the amount of this payoff is £1 if the stock price \( S_T \) at time \( T \) is at least \( a \), for some positive number \( a \), and zero otherwise. Let \( S \) be the price of the stock and assume, as usual, that \( S \) follows the process

\[
dS = \mu Sdt + \sigma SdB
\]

for constants \( \mu \) and \( \sigma > 0 \) and where \( B \) is a Brownian motion. Assume further that all interest rates are constant and equal to \( r \).

(a) Use Ito’s Lemma to show that \( \log S \) follows the process

\[
d(\log S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.
\]

(b) Find an expression for the probability in a risk-neutral world of the event \( S_T \leq a \).

(c) Apply a risk-neutral valuation argument to show that, for any \( 0 \leq t \leq T \), the value of this derivative equals

\[
e^{-r(T-t)} \Phi \left( \frac{\log (S_t/a) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma \sqrt{T-t}} \right),
\]

where \( \Phi \) is the cumulative distribution function of the standard normal distribution.

(ii) (a) Verify that \( f(S, t) = e^{(2r+3\sigma^2)(T-t)}S^3 \) is a solution of the Black-Scholes partial differential equation

\[
\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.
\]

(b) Consider a derivative with underlying asset whose price \( S \) follows the Ito process \( dS = \mu Sdt + \sigma SdB \) and which provides a single payoff at time \( T > 0 \) in the amount of \( S_T^3 \), where \( S_T \) is the underlying asset price at time \( T \). What is the price of this derivative at time \( 0 \leq t < T \)?
(i) Define the following concepts in the context of Portfolio Theory.

(a) The market portfolio.  

(b) The capital market line.  

(c) The beta-coefficient of an investment.  

(ii) Consider a market with risk-free return \( r_B \) and whose market portfolio \( M \) has expected return \( r_M \) and standard deviation of returns \( \sigma_M \). Let \( A \) be an investment with expected return of \( r_A \), standard deviation of returns \( \sigma_A \) and beta coefficient \( \beta \).

(a) What is the slope of the capital market line?  

(b) Show that the market portfolio is the unique portfolio \( P \) which maximizes

\[
\frac{r_P - r_B}{\sigma_P}
\]

as \( P \) ranges over all portfolios consisting entirely of risky investments.  

(c) Describe parametrically the curve \( c \) in the \( \sigma-r \) plane consisting of all points corresponding to investments spread between \( A \) and \( M \).  

(d) Explain why \( c \) is tangent to the capital market line at the point \( M \).  

(e) Use (d) to show that

\[
r_A = \beta(r_M - r_B) + r_B.
\]

End of Question Paper