SCHOOL OF MATHEMATICS AND STATISTICS

Financial Mathematics

2 hours and 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
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Registration number from U-Card (9 digits)
to be completed by student

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Consider a perpetual bond that pays £5 once a year, every year, and whose first payment occurs in 6 months. Assume that spot interest rates for all maturities are 3%.

(i) Find the price of the bond. \((8 \text{ marks})\)

(ii) Consider a \(N\)-year forward contract on the perpetual bond, where \(N\) is a positive integer. Show that the correct forward price in this forward contract is identical to the spot price of the bond. \((8 \text{ marks})\)

(iii) You are given the opportunity to take a long position in a two-year forward contract as in (ii) at a forward price of £160. Describe in detail an arbitrage opportunity available to you. \((9 \text{ marks})\)

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(i) (a) Describe a portfolio consisting entirely of European put options on the same stock, with same expiration time \(T > 0\), but with different strike prices, and whose payoff at time \(T\) as a function of \(S\), the spot price of the stock at time \(T\), is described by the graph below. \((4 \text{ marks})\)

\begin{center}
\begin{tikzpicture}
\begin{axis}[
    axis lines=left,
    xlabel={$S$},
    ylabel={Payoff},
    xmin=-1, xmax=80,
    ymin=-1, ymax=35,
    ytick={0,10,20,30},
    yticklabels={-10,10,20,30},
    xtick={10,20,30,40,50,60,70},
    xticklabels={10,20,30,40,50,60,70},
]
\addplot[domain=10:70, samples=100, black, thick] {max(0, -10 + 10 + 5 + 20)};
\end{axis}
\end{tikzpicture}
\end{center}

(b) Let \(p_{20}, p_{30}, p_{40}\) and \(p_{60}\) be the prices of the above put options with strike prices 20, 30, 40 and 60, respectively, and let \(c_{10}\) be the price of a European call option on the same stock, with expiration at time \(T\) and with strike price 10. By comparing the payoff of the portfolio in (a) and the payoff of the call option above, describe an inequality involving \(c_{10}, p_{20}, p_{30}, p_{40}\) and \(p_{60}\). \((5 \text{ marks})\)

(ii) Describe what American put options are. \((3 \text{ marks})\)

(iii) The price of a stock which pays no dividends is currently £20. Over each of the next three 1-year periods the stock price will either increase by 10% or decrease by 10%. Suppose that all interest rates are constant and equal to 3%.

(a) Use a binomial tree to find the price of a three-year American put option on this stock with strike price £20. \((11 \text{ marks})\)

(b) Describe all circumstances when a rational investor should exercise the option described in (a) before its expiration. \((2 \text{ marks})\)
Consider a derivative on a stock which entitles the holder to one payoff at time $T$; the amount of this payoff is £1 if the stock price $S_T$ at time $T$ is at least $a$, for some positive number $a$, and zero otherwise. Let $S$ be the price of the stock and assume, as usual, that $S$ follows the process

$$dS = \mu Sdt + \sigma SdB$$

for constants $\mu$ and $\sigma > 0$ and where $B$ is a Brownian motion. Assume further that all interest rates are constant and equal to $r$.

(a) Use Ito's Lemma to show that $\log S$ follows the process

$$d(\log S) = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dB.$$

(6 marks)

(b) Find an expression for the probability in a risk-neutral world of the event $S_T \leq a$. 

(8 marks)

(c) Apply a risk-neutral valuation argument to show that, for any $0 \leq t \leq T$, the value of this derivative equals

$$e^{-r(T-t)} \Phi \left( \frac{\log(S_t/a) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right),$$

where $\Phi$ is the cumulative distribution function of the standard normal distribution. 

(3 marks)

(ii) (a) Verify that $f(S,t) = e^{(2r + 3\sigma^2)(T-t)} S^3$ is a solution of the Black-Scholes partial differential equation

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf.$$

(4 marks)

(b) Consider a derivative with underlying asset whose price $S$ follows the Ito process $dS = \mu Sdt + \sigma SdB$ and which provides a single payoff at time $T > 0$ in the amount of $S_T^a$, where $S_T$ is the underlying asset price at time $T$. What is the price of this derivative at time $0 \leq t < T$?

(4 marks)
4 (i) Define the following concepts in the context of Portfolio Theory.

(a) The market portfolio. (2 marks)
(b) The capital market line. (2 marks)
(c) The beta-coefficient of an investment. (2 marks)

(ii) Consider a market with risk-free return $r_B$ and whose market portfolio $M$ has expected return $r_M$ and standard deviation of returns $\sigma_M$. Let $A$ be an investment with expected return of $r_A$, standard deviation of returns $\sigma_A$ and beta coefficient $\beta$.

(a) What is the slope of the capital market line? (2 marks)

(b) Show that the market portfolio is the unique portfolio $P$ which maximizes

$$\frac{r_P - r_B}{\sigma_P}$$

as $P$ ranges over all portfolios consisting entirely of risky investments. (3 marks)

(c) Describe parametrically the curve $c$ in the $\sigma$-$r$ plane consisting of all points corresponding to investments spread between $A$ and $M$. (6 marks)

(d) Explain why $c$ is tangent to the capital market line at the point $M$. (3 marks)

(e) Use (d) to show that

$$r_A = \beta(r_M - r_B) + r_B.$$ (5 marks)

End of Question Paper