Section A

A1 Solve the following inequalities:
   (i) \(|3x - 3| < 9|; \quad (2 \text{ marks})
   (ii) \(|x^2 + x - 6| > 0. \quad (2 \text{ marks})

A2 Evaluate the following limits:
   (i) \(\lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 3x + 2}; \quad (2 \text{ marks})
   (ii) \(\lim_{x \to \pi/2} \frac{2x^2 - 3\pi x + \pi^2}{\cos x}. \quad (2 \text{ marks})

A3 Give the equation of the tangent plane to the surface \(z = 5x^2 + 3xy + y^2\) at the point \((1,1,9).\) \quad (3 \text{ marks})

A4 Let \(f(x,y) = \frac{x - y}{\sqrt{x^2 - y^2}}.\) Evaluate \(f\) at the point \((5,4).\) Using partial derivatives, approximate the change \(\delta f\) when we move to the nearby point \((5.1, 3.9).\) \quad (5 \text{ marks})
A5  Let $z$ be a function of $u$ and $v$ where $u = x - y$ and $v = 2x + 2y$.

(i) Show that

\[ \frac{\partial z}{\partial x} \frac{\partial z}{\partial y} = 4 \left( \frac{\partial z}{\partial v} \right)^2 - \left( \frac{\partial z}{\partial u} \right)^2. \]

(2 marks)

(ii) Assuming equality of mixed second-order partial derivatives, show that

\[ \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 2 \frac{\partial^2 z}{\partial u^2} + 8 \frac{\partial^2 z}{\partial v^2}. \]

(3 marks)

A6  Let $R$ be the triangular region in the $(x, y)$-plane with vertices $(0, 0)$, $(2, 0)$ and $(0, 1)$. Calculate the integral

\[ \int \int_R 2xy + 6x^2y \, dx \, dy. \]

(6 marks)

A7  What is the radius of convergence of

\[ \sum_{n=1}^{\infty} \frac{(3n)!}{(n!)^3 5^n} z^n? \]

(3 marks)

Section B

B1  Consider the following system of linear equations:

\[
\begin{cases}
  x + 2y + 7z = 3, \\
  -2x + 5y + 4z = 3, \\
  -5x + 6y - 3z = 1.
\end{cases}
\]

By Gauss-Jordan elimination, put the relevant augmented matrix into a row-echelon form and solve the equations. (4 marks)

B2  Let $A, B$ be $2 \times 2$ matrices. For each statement below, decide whether it is true or false. Prove if it is true and give a counter-example if it is false.

(i) If $AB = 0$ and $A \neq 0$, then $B = 0$. (2 marks)

(ii) If $AB = 0$ and $\det(A) \neq 0$, then $B = 0$. (2 marks)
B3  By performing elementary row operations, show that

\[
\begin{vmatrix}
1 & 1 & 1 \\
x & y & z \\
x^3 & y^3 & z^3
\end{vmatrix} = (x - y)(y - z)(z - x)(x + y + z).
\]

Hint: \(x^3 - y^3 = (x - y)(x^2 + xy + y^2)\).

(6 marks)

B4  Consider a family of curves on a plane which has two real parameters \(u\) and \(v\)

\[
x = c \cosh u \cos v, \quad y = c \sinh u \sin v,
\]

where \(c > 0\) is a constant and \(0 \leq v \leq 2\pi\).

(i) By assuming that \(u\) is a constant, derive an equation of an ellipse

\[
\frac{x^2}{c^2 \cosh^2 u} + \frac{y^2}{c^2 \sinh^2 u} = 1.
\]

(2 marks)

(ii) By assuming that \(v\) is a constant, derive an equation of a hyperbola.

(2 marks)

(iii) Show that the two conics share their foci in common.

(2 marks)

(iv) Show that the two conics are perpendicular at each point of intersection, that is, their tangents are orthogonal to each other.

(2 marks)

Note: \(\cosh x = \frac{e^x + e^{-x}}{2}, \ \sinh x = \frac{e^x - e^{-x}}{2}\).

B5  We prove that the points \((p, q)\) which are the intersection of two orthogonal tangent lines from an ellipse lie on a circle.

(i) Let \(E\) be the ellipse given by

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]

where \(a, b > 0\). Let \(p\) and \(q\) be any real numbers. Find a quadratic equation in \(m\) which holds when a straight line through \((p, q)\) with gradient \(m\) is a tangent to \(E\). (You are encouraged to draw a picture.)

(4 marks)

(ii) By considering the two solutions for \(m\) above, show that the points \((p, q)\) for which the two tangent lines to \(E\) that pass through \((p, q)\) are orthogonal form a circle.

(4 marks)

End of Question Paper