SCHOOL OF MATHEMATICS AND STATISTICS

Applied Differential Equations

2 hours

Attempt all FOUR questions.

1 (i) Show that the following linear multi-step method

\[ y_{n+1} = y_n - \frac{h}{12}(5f_{n+1} + 31f_n - 17f_{n-1} + 5f_{n-2}), \]

is consistent and stable, where \( h \) is the step-size, \( f_n = f(x_n, y_n) \) and \( y'(x) = f(x, y) \).

(ii) The following table contains grid-point values of two solutions \( Y_1(x) \) and \( Y_2(x) \) of a linear differential equation \( \frac{d^2y}{dx^2} = f(x,y,y') \) obtained using the fourth-order Runge-Kutta method.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( Y_1(x) )</th>
<th>( Y_2(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>1.07444</td>
<td>1.36222</td>
</tr>
<tr>
<td>2.0</td>
<td>1.36280</td>
<td>2.05472</td>
</tr>
<tr>
<td>2.5</td>
<td>2.02865</td>
<td>3.35550</td>
</tr>
<tr>
<td>3.0</td>
<td>3.41281</td>
<td>5.85592</td>
</tr>
</tbody>
</table>

\( Y_1(x) \) was determined using the initial conditions \( y(1) = 1, \ y'(1) = 0 \), and \( Y_2(x) \) was obtained using \( y(1) = 1, \ y'(1) = 0.5 \). Form a linear combination of these two solutions which is the numerical solution to the equation \( \frac{d^2y}{dx^2} = f(x,y,y') \) with boundary conditions \( y(1) = 1, \ y(3) = 2 \). Calculate the values of this solution at all the \( x \)-values given in the table.

(iii) The Taylor series expansion for \( y(x_{n+1}) \), where \( x_{n+1} = x_n + h \), is given as

\[ y(x_{n+1}) = y(x_n) + hy'(x_n) + \frac{h^2}{2}y''(x_n) + \frac{h^3}{6}y'''(x_n) + O(h^4). \]

Write down the expansion for \( y(x_{n-1}) \), where \( x_{n-1} = x_n - h \), and then derive the relation

\[ y''(x_n) = \frac{y(x_{n+1}) - 2y(x_n) + y(x_{n-1})}{h^2} + O(h^2). \]
2 A single step method for the equation \( y' = f(x, y) \) is defined by the following formulas:

\[
k_1 = hf_n, \quad k_2 = hf \left( x_n + \frac{5}{6} h, y_n + \frac{5}{6} k_1 \right),
\]

\[
y_{n+1} = y_n + \frac{2}{5} k_1 + \frac{3}{5} k_2,
\]

where \( f_n = f(x_n, y_n) \).

(i) Write down its local discretisation error \( \tau(x_n, h) \). By finding the limit of \( \tau(x_n, h) \) when \( h \to 0 \), show that the method is consistent. \( (8 \text{ marks}) \)

(ii) We want to use the method to solve the equation

\[
y'(x) = -0.2y(x), \quad y(0) = 1.
\]

Find the largest possible step-size \( h \) in order that the method is absolutely stable. \( (10 \text{ marks}) \)

(iii) Two approximate solutions at \( x = 3 \) have been found using the above method with two step-sizes. The results are:

<table>
<thead>
<tr>
<th>( h )</th>
<th>Approximate solutions for ( y(3) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.3</td>
<td>0.5490</td>
</tr>
<tr>
<td>0.6</td>
<td>0.5497</td>
</tr>
</tbody>
</table>

Without using the analytic solution of Equation (1), use the data in the table to estimate the step-size required to ensure that the absolute value of the global discretization error in \( y(3) \) is smaller than \( 10^{-5} \). You may assume that the above method is of order 2. Work throughout with four decimal places. \( (5 \text{ marks}) \)

(iv) Estimate the magnitude of the global discretisation error of the solution at \( x = 3 \) if the step-size is \( h = 0.9 \). \( (2 \text{ marks}) \)
(i) Consider Laplace’s equation

\[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \]

for \(0 \leq x \leq 1\) and \(0 \leq y \leq 1\), with boundary conditions

\[ u(0, y) = 0, \quad -u(x, 0) + 2 \frac{\partial u(x, y)}{\partial y} \bigg|_{y=0} = 0, \quad u(x, 1) = 0, \]

and \(u(1, y)\) unspecified.

(a) Letting \(u(x, y) = X(x)Y(y)\) be a separable solution for the above equation, show that

\[ \frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \alpha. \]

Explain why \(\alpha\) must be a constant. \(\text{(3 marks)}\)

(b) Given that there is only a trivial separable solution when \(\alpha \leq 0\), show that the non-trivial solution has the following expression

\[ u(x, y) = C \sinh(sx)[2s \cos(sy) + \sin(sy)], \]

where \(C\) is a constant and \(s\) is a root of the following algebraic equation

\[ 2s \cos(s) + \sin(s) = 0. \]

\(\text{(16 marks)}\)

(ii) Show that, with variable substitutions \(\eta = x + ct\), \(\nu = x - ct\), the inhomogeneous wave equation

\[ \frac{\partial^2 u}{\partial \eta^2} = c^2 \frac{\partial^2 u}{\partial x^2} + g \]

becomes

\[ \frac{\partial^2 u}{\partial \eta \partial \nu} = -\frac{g}{4c^2}, \]

where \(g\) is a constant. \(\text{(6 marks)}\)
Consider the following equation for $u(x, t)$

$$\frac{\partial u}{\partial t} + 2 \frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2},$$

for $0 \leq x \leq 1$ and $t \geq 0$, subject to inhomogeneous boundary conditions

$$u(0, t) = 1, \quad u(1, t) = e^2,$$

and initial condition

$$u(x, 0) = e^x \sin(2\pi x) + e^{2x}.$$

(i) Letting $u(x, t) = v(x, t) + e^2x$, find the equation for $v(x, t)$, and its boundary and initial conditions. \hspace{1cm} \text{(6 marks)}

(ii) Use separation of variables to find the solution for $u(x, t)$. You may use the following result: If $a^2 - 4b \geq 0$, and the boundary conditions $X(0) = X(1) = 0$ are imposed, then the equation

$$X''(x) + aX'(x) + bX(x) = 0$$

has only the trivial solution. \hspace{1cm} \text{(19 marks)}

End of Question Paper