Answer all four questions.

You should justify your answers carefully unless the question states otherwise.

1. (i) Write down the units in $\mathbb{Z}_{20}$, justifying your answer. Is the multiplicative group they form cyclic? (7 marks)

(ii) Is it true that the units in the ring $\mathbb{Z}_{20}[x]$ are the same as those in $\mathbb{Z}_{20}$, expressed as constant polynomials? Prove this or give a counterexample. (4 marks)

(iii) Let $R$ be a ring. Prove that an element $r$ of $R$ cannot be both a unit and a zero-divisor. (4 marks)

Additional marks for rigour and presentation. (5 marks)

2. (i) Let $d$ be a square-free integer with $d \neq 1$. Recall that the norm of an element $r = a + b\sqrt{d}$ of $\mathbb{Z}[\sqrt{d}]$, where $a, b \in \mathbb{Z}$, is given by

$$N(a + b\sqrt{d}) = |a^2 - b^2d|.$$ 

Show that $\mathbb{Z}[^5\sqrt{-5}]$ has no element of norm 7. Hence show that any element of norm 49 is irreducible. (6 marks)

(ii) By considering the element $2 + 3\sqrt{-5}$ in $\mathbb{Z}[\sqrt{-5}]$, express 49 as a product of irreducible factors in two different ways, and deduce that $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorisation domain. You may use that fact that an element of $\mathbb{Z}[\sqrt{-5}]$ is a unit if and only if its norm is 1. Justify your answer. (6 marks)

(iii) Let $R$ be a unique factorisation domain and let $S$ be a subring of $R$. Is $S$ necessarily a unique factorisation domain? Justify your answer briefly. (3 marks)

Additional marks for rigour and presentation. (5 marks)
Let $G$ be a group of order 55 with trivial centre.

(i) Find the class equation of $G$, justifying your answer. (3 marks)

(ii) What are the possible orders of elements of $G$? How many elements of each order are there? (7 marks)

(iii) Let $h \in G$ be an element of order 11. Let $H = \langle h \rangle$, the subgroup generated by $h$. Use the class equation to show that $H$ is a normal subgroup of $G$. Show that $H$ is the only normal subgroup of $G$, other than the trivial group $\{e\}$ and $G$ itself. (5 marks)

Additional marks for rigour and presentation. (5 marks)

Consider the square divided into regions labelled 1, 2, 3 and 4, and lines of symmetry labelled $a$, $b$, $c$ and $d$, as below.

Recall that $D_4$ is the group of symmetries of the square, and write $e$ for the identity of the group, $r$ for rotation through $\frac{\pi}{2}$ anti-clockwise, and $s_i$ for reflection in the line $i$, for $i = a, b, c, d$. Observe that $D_4$ acts on the numbered regions of the square, inducing a homomorphism $f : D_4 \rightarrow S_4$.

(i) List the elements of $D_4$, and write down the effect of $f$ on each of them. (6 marks)

(ii) Write down the kernel and image of the homomorphism $f$. Is the image of $f$ isomorphic to the cyclic group of order 4 or the Klein 4-group? Is $f$ injective? Is $f$ surjective? (4 marks)

(iii) State, without proof, the First Isomorphism Theorem for groups. (2 marks)

(iv) What does the First Isomorphism Theorem tell us in this case? (3 marks)

Additional marks for rigour and presentation. (5 marks)

End of Question Paper