Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

Throughout the paper $I(a, b)$ denotes the open interval $\{t \in \mathbb{R} \mid a < t < b\}$ and $I$ denotes an open interval in $\mathbb{R}$ with unspecified endpoints.

1. (i) Define $\varphi: \mathbb{R}^4 \to \mathbb{R}^3$ by $\varphi(x, y, z, t) = (u, v, w)$, where
   \[ u = 2(xt - yz), \quad v = 2(xy + zt), \quad w = x^2 - y^2 + z^2 - t^2. \]

   Also define $F: \mathbb{R}^3 \to \mathbb{R}$ by $F(u, v, w) = u^2 + v^2 + w^2$.
   
   (a) Find the composite $F \circ \varphi: \mathbb{R}^4 \to \mathbb{R}$, simplifying your answer as much as possible.
   
   (b) Let $S = \{(u, v, w) \in \mathbb{R}^3 \mid u^2 + v^2 + w^2 = 1\}$. Find $\varphi^{-1}(S)$.  

   (8 marks)

(ii) Let $(a, b) \in \mathbb{R}^2$ and let $r > 0$. Define the open ball $B((a, b), r)$.

   Define what it means for a set $M \subseteq \mathbb{R}^2$ to be open.  

   (5 marks)

   Let $(a, b) \in \mathbb{R}^2$ and define $M = \mathbb{R}^2 \setminus \{(a, b)\}$. Prove, directly from your definition, that $M$ is an open set.  

   (5 marks)

(iii) Define $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ by $\varphi(x, y) = (y, xy)$. Find the image of $\varphi$ and show that it is not an open set.  

   (7 marks)
2 (i) (a) Define what it means for a function $F: M \to \mathbb{R}$, where $M \subseteq \mathbb{R}^2$ is an open set, to be continuous at $(a, b) \in M$.

(b) Define what it means for a function $F: M \to \mathbb{R}$, where $M \subseteq \mathbb{R}^2$ is an open set, to be $C^1$. (7 marks)

(ii) (a) Let $H: M \to \mathbb{R}$ be a continuous function defined on an open set $M \subseteq \mathbb{R}^2$ and let $(a, b) \in M$.

If $H(a, b) \neq 0$ show that there is an $r > 0$ such that $B((a, b), r) \subseteq M$ and $H(x, y) \neq 0$ for all $(x, y) \in B((a, b), r)$. (5 marks)

(b) Let $F: M \to \mathbb{R}$ and $G: M \to \mathbb{R}$ be $C^1$ functions defined on an open set $M \subseteq \mathbb{R}^2$. Let $(a, b) \in M$ be such that

\[
\begin{vmatrix}
\frac{\partial F}{\partial x}(a, b) & \frac{\partial F}{\partial y}(a, b) \\
\frac{\partial G}{\partial x}(a, b) & \frac{\partial G}{\partial y}(a, b)
\end{vmatrix} \neq 0.
\]

Show that there is $r > 0$ such that $B((a, b), r) \subseteq M$ and

\[
\begin{vmatrix}
\frac{\partial F}{\partial x}(x, y) & \frac{\partial F}{\partial y}(x, y) \\
\frac{\partial G}{\partial x}(x, y) & \frac{\partial G}{\partial y}(x, y)
\end{vmatrix} \neq 0
\]

for all $(x, y) \in B((a, b), r)$. (7 marks)

(iii) Define $F: \mathbb{R}^2 \to \mathbb{R}$ by

\[
F(x, y) = \begin{cases} 
\frac{x^2 y}{x^4 + y^2} & \text{for } (x, y) \neq (0, 0) \\
0 & \text{for } (x, y) = (0, 0).
\end{cases}
\]

Show that $F$ is not continuous at $(0, 0)$, stating clearly any theorem from the course that you use. (6 marks)

3 (i) State the triangle identity for elements of $\mathbb{R}^2$. (3 marks)

(ii) Let $F: M \to \mathbb{R}$ and $G: M \to \mathbb{R}$ be functions defined on an open set $M \subseteq \mathbb{R}^2$ which are continuous at a point $(a, b) \in M$.

Prove that $FG$ is continuous at $(a, b)$. (14 marks)

(iii) Let $m, n, p$ and $q$ be non-zero integers, and let $\lambda \in \mathbb{R}$, $\lambda \neq 0$.

Write $M = \{(x, y) \mid x > 0, y > 0\}$ and define $\varphi: M \to \mathbb{R}^2$ by

\[
\varphi(x, y) = (\lambda x^m y^n, \lambda x^p y^q).
\]

For which values of $m, n, p, q$ and $\lambda$ is the determinant of the derivative of $\varphi$ equal to 1 for all $(x, y) \in M$? (8 marks)
(i) Let $u, v \in \mathbb{R}$ and consider the cubic equation

$$t^3 + ut + v = 0. \tag{*}$$

Suppose that $(*)$ has three real roots $x, y, z,$ some or all of which may be equal.

(a) Prove that

$$u = -x^2 - xy - y^2, \quad v = x^2y + xy^2.$$  \hspace{1cm} (3 marks)

(b) Define $\varphi: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\varphi(x, y) = (-x^2 - xy - y^2, x^2y + xy^2).$$

Determine the set $E$ of points $(x, y)$ at which $\det D(\varphi)(x, y) = 0.$  \hspace{1cm} (4 marks)

(c) Determine the image $\varphi(E)$ of $E$ under $\varphi,$ expressing your answer in terms of $u$ and $v.$ \hspace{1cm} (4 marks)

(ii) Let $I \subseteq \mathbb{R}$ be an open interval and let $M \subseteq \mathbb{R}^3$ be an open set. Let $\alpha: I \to \mathbb{R}^3$ and $\varphi: M \to \mathbb{R}^2$ be $C^1$ maps such that $\alpha(t) \in M$ for all $t \in I.$

(a) Write down (but do not prove) the Chain Rule for $\varphi \circ \alpha.$

(b) Now assume that $D(\varphi \circ \alpha)(t) = (0, 0)$ for all $t \in I$ and that there is a point $t_0 \in I$ such that

$$\det \begin{bmatrix} \frac{\partial \varphi_1}{\partial x}(x_0, y_0, z_0) & \frac{\partial \varphi_1}{\partial y}(x_0, y_0, z_0) \\ \frac{\partial \varphi_2}{\partial x}(x_0, y_0, z_0) & \frac{\partial \varphi_2}{\partial y}(x_0, y_0, z_0) \end{bmatrix} \neq 0$$

where $(x_0, y_0, z_0) = \alpha(t_0)$ and $\varphi = (\varphi_1, \varphi_2).$

Express $\alpha_1'(t_0)$ and $\alpha_2'(t_0)$ in terms of partial derivatives of $\varphi$ and $\alpha_3'(t_0).$ \hspace{1cm} (14 marks)
(i) State the Chain Rule as it applies to a $C^1$ map $\psi: M \to N$ and a $C^1$ function $H: N \to \mathbb{R}$, where $M$ and $N$ are open subsets of $\mathbb{R}^2$. (3 marks)

(ii) Let $M$ and $N$ be open subsets of $\mathbb{R}^3$. Define what it means for a map $\varphi: M \to N$ to be a diffeomorphism. (3 marks)

(iii) State carefully and in full the Local Diffeomorphism Theorem for maps $M \to N$ where $M$ and $N$ are open sets in $\mathbb{R}^3$. (6 marks)

(iv) Let $S: \mathbb{R}^2 \to \mathbb{R}$ be a $C^1$ function, written as $S(x,u)$, for which the two partial derivatives

$$
H(x,u) = \frac{\partial S}{\partial x}(x,u), \quad \text{and} \quad K(x,u) = \frac{\partial S}{\partial u}(x,u),
$$

are also $C^1$. You may use without proof the fact that $\frac{\partial H}{\partial u} = \frac{\partial K}{\partial x}$ everywhere.

Now suppose there is a point $(x_0, u_0) \in \mathbb{R}^2$ at which $\frac{\partial^2 S}{\partial u \partial x}(x_0, u_0) \neq 0$.

Define $\tilde{H}: \mathbb{R}^2 \to \mathbb{R}^2$ by

$$
\tilde{H}(x,u) = (x, H(x,u)).
$$

Show that the Local Diffeomorphism Theorem can be applied to $\tilde{H}$ at $(x_0, u_0)$, stating your conclusion in full.

Deduce that there is an open set $N_1 \subseteq \mathbb{R}^2$ and a $C^1$ function $F: N_1 \to \mathbb{R}$, written $u = F(x,y)$, such that

$$
H(x, F(x,y)) = y
$$

for all $(x,y) \in N_1$. (8 marks)

(v) Using (i), or otherwise, show that

$$
\frac{\partial H}{\partial x} + \frac{\partial H}{\partial u} \frac{\partial F}{\partial x} = 0, \quad \frac{\partial H}{\partial u} \frac{\partial F}{\partial y} = 1.
$$

(5 marks)

End of Question Paper