Consider a flow between parallel plates $y = 0, h$. The upper plate $y = h (> 0)$ is moving with a constant speed $V (> 0)$ in the $x$-direction and the lower plate is at rest. We look for a steady flow of the form

$$\mathbf{v} = (u(x, y), 0, 0).$$

This is a two-dimensional problem and no quantities depend on $z$. We assume that the pressure does not depend on $x$.

(i) By using the continuity equation, show that $u$ is actually a function of $y$ only. (2 marks)

(ii) Use the Navier-Stokes equations to show that

$$\frac{d^2 u}{dy^2} = 0 \quad \text{and} \quad \frac{dp}{dy} = 0.$$

(8 marks)

(iii) Determine $u$ by taking into account the boundary conditions. (4 marks)

(iv) Compute the wall shear stress at $y = 0$ and confirm its physical dimension is consistent. (5 marks)

(v) Compute the vorticity of the flow and state in which direction it points. (3 marks)

(vi) Compute the volume flux (that is, the net volume passing between the plates per unit time per unit width) of the flow. (3 marks)
(i) Suppose that an incompressible velocity field in an inviscid fluid is given by

\[ \mathbf{v} = (\alpha x - \Omega y, -y + \Omega x, 0) \]

relative to Cartesian coordinates \((x, y, z)\), where \(\alpha\) and \(\Omega\) may be functions of time. Find \(\alpha\) and the vorticity, and show that \(\Omega\) must be constant with respect to time. \(7\) marks

(ii) The material derivative of vector \(\mathbf{A}\) is defined as

\[ \frac{D\mathbf{A}}{Dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{A}, \]

where \(\mathbf{u}\) denotes fluid velocity which is in general compressible.

If \(\theta(x, t)\) is a scalar function of position and time show that in an inviscid fluid

\[ \frac{D}{Dt}(\omega \cdot \nabla \theta) = (\omega \cdot \nabla) \frac{D\theta}{Dt}. \]

Hence deduce that if \(\theta(x, t)\) is any scalar quantity which is conserved by all fluid elements moving with the flow, the \((\omega \cdot \nabla)\theta\) is also a constant. \(8\) marks

(iii) Using the result in (iii), prove the following acceleration formula

\[ \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \nabla \left( \frac{|\mathbf{u}|^2}{2} \right) + (\nabla \times \mathbf{u}) \times \mathbf{u}. \]

\(10\) marks
Consider the two-dimensional Navier-Stokes equations and the continuity equation for an incompressible velocity field \( \mathbf{u} = (u, v) \),

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right),
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right),
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,
\]

where \( \nu \) denotes kinematic viscosity, \( p \) the pressure and \( \rho \) uniform density.

(i) Show that the vorticity \( \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \) satisfies

\[
\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right).
\]

(9 marks)

(ii) Using a stream function \( \psi \) such that

\[
\mathbf{u} = \left( \frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right)
\]

show that the above equation can be written as

\[
\frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu \left( \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right),
\]

where

\[
J(\omega, \psi) = \frac{\partial \omega}{\partial x} \frac{\partial \psi}{\partial y} - \frac{\partial \omega}{\partial y} \frac{\partial \psi}{\partial x}
\]

denotes a Jacobian determinant. (3 marks)

(iii) Show that for \( \omega = F(\psi) \) with an arbitrary function \( F \), the advection term vanishes. (3 marks)

(iv) Consider

\[
\psi = A(t) \cos \left( \frac{\pi x}{l} \right) \cos \left( \frac{\pi y}{l} \right),
\]

where \( l(>0) \) is a constant length scale. Show that we can choose a function \( F(\psi) \) as defined in (iii). Determine \( A(t) \) in such a way that it is a solution of the two-dimensional Navier-Stokes equations. (10 marks)
A fluid moves in a steady two-dimensional flow in the region defined by 
\( x \geq 0, \; y \geq 0 \). The boundary with equation \( y = 0 \) is occupied by a 
stationary flat plate. Given that the \( x \)-component of velocity \( u \to U \) as 
\( y \to \infty \), where \( U \) is a constant, write down the expressions for 
(a) the displacement thickness \( \delta_1 \) of the boundary layer, 
and 
(b) the momentum thickness \( \delta_2 \) of the boundary layer. 

\[ (4 \text{ marks}) \]

(c) When the flow is given approximately by 
\[ \frac{u}{U} = \left( \frac{y}{\delta} \right)^m \] 
where \( 0 < m < 1 \), 
compute \( \delta_1 \) and \( \delta_2 \) explicitly and hence show which is the largest. 

\[ (6 \text{ marks}) \]

(ii) Consider a steady two-dimensional flow past a semi-infinite solid boundary 
along \( y = 0 \) in the region \( y \geq 0, \; x \geq 0 \). Blasius's boundary layer equations 
for it are given by 
\[ u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2}, \]
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \]

Derive the energy equation for the boundary layer 
\[ \frac{d}{dx} \int_0^\infty \frac{u}{U} \left( 1 - \frac{u^2}{U^2} \right) dy = \frac{2D}{\rho U^3}, \]
where \( u, \; v \) are the \( x \)- and \( y \)- components of the velocity and 
\[ D = \mu \int_0^\infty \left( \frac{\partial u}{\partial y} \right)^2 dy \]
is the dissipation rate of energy. The constant \( U \) is the \( x \)-component of the 
velocity as \( y \to \infty \). 

\[ (15 \text{ marks}) \]

End of Question Paper