Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

1. Use the two-phase method to solve the following problem:

   \[
   \text{Min } z = 2x_1 + 3x_2 - 5x_3 \\
   x_1 + x_2 + x_3 = 7 \\
   2x_1 - 5x_2 + x_3 \geq 10
   \]

   where \(x_1, x_2\) and \(x_3\) are non-negative. Clearly state your optimal solution and optimal cost. (Hint: Not counting the pre-processing steps, you should be able to finish phase 1 with two simplex iterations, and phase 2 with one simplex iteration.)

   \((25 \text{ marks})\)
Bill wants to celebrate his new job by watching every movie showing in cinemas in five nearby towns. If he travels to another town, he will stay there until he has watched all the movies on show there. The following table provides the information about the 9 movies on show and the towns:

<table>
<thead>
<tr>
<th>Location</th>
<th>Movies</th>
<th>Round-trip miles</th>
<th>Cost per movie (£)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Town 1</td>
<td>1, 3, 4</td>
<td>0</td>
<td>7.95</td>
</tr>
<tr>
<td>Town 2</td>
<td>1, 6, 8</td>
<td>25</td>
<td>5.50</td>
</tr>
<tr>
<td>Town 3</td>
<td>2, 5, 6, 7</td>
<td>30</td>
<td>5.00</td>
</tr>
<tr>
<td>Town 4</td>
<td>1, 8, 9</td>
<td>28</td>
<td>7.00</td>
</tr>
<tr>
<td>Town 5</td>
<td>2, 4, 5, 7</td>
<td>40</td>
<td>4.95</td>
</tr>
</tbody>
</table>

The cost of driving is 75 pence per mile. Bill wishes to determine the towns he needs to visit to see all the movies while minimising his total cost.

Defining binary variables \( x_i \) such that

\[
x_i = \begin{cases} 
1 & \text{if Bill visits town } i \\
0 & \text{if Bill does NOT visit town } i 
\end{cases}
\]

for \( i = 1, 2, ..., 5 \), formulate the mixed integer/linear programming problem. Do NOT solve the resulting problem. \( (18 \text{ marks}) \)

(ii) For each of the following sets of conditions, set up the appropriate mathematical expressions suitable for integer linear programming:

(a) If \( g^T x > c \), then \( f^T x \geq d \) where \( g \) and \( f \) are two constant vectors, and \( c \) and \( d \) are constants.

(b) Either \( x_1 + 2x_2 \leq 6 \) is imposed or (both \( 2x_1 + x_2 \leq 8 \) and \( x_1 - x_2 \geq 3 \) are imposed). \( (7 \text{ marks}) \)
A company produces units of A, B and C, using labour and raw materials $M_1$ and $M_2$. The requirements for per unit of each product are given as follows:

<table>
<thead>
<tr>
<th></th>
<th>$M_1$(kg)</th>
<th>$M_2$(kg)</th>
<th>Labour(hour)</th>
<th>Profit(£)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3</td>
<td>8</td>
<td>2</td>
<td>69</td>
</tr>
<tr>
<td>B</td>
<td>6</td>
<td>12.5</td>
<td>3</td>
<td>111</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>3.5</td>
<td>1</td>
<td>42</td>
</tr>
</tbody>
</table>

The company has available in the next month 1 ton of $M_1$, 2.5 tons of $M_2$, and 550 hours of labour. To determine the optimal production schedule, the manager defines $x_1$, $x_2$ and $x_3$ as the number of A’s, B’s, and C’s to be produced in the next month, and formulates the following model for the optimal profit:

Maximise  
$$ z = 69x_1 + 111x_2 + 42x_3 $$  
subject to
$$ 3x_1 + 6x_2 + 2x_3 \leq 1000 $$  
$$ 8x_1 + 12.5x_2 + 3.5x_3 \leq 2500 $$  
$$ 2x_1 + 3x_2 + x_3 \leq 550 $$

Using slack variables $x_4$, $x_5$, and $x_6$, the optimal tableau is found as follows:

<table>
<thead>
<tr>
<th>Basis</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z$</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>15</td>
<td>0</td>
<td>12</td>
<td>21600</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>3</td>
<td>1</td>
<td>2</td>
<td>0</td>
<td>-3</td>
<td>350</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>-5.5</td>
<td>475</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
<td>100</td>
</tr>
</tbody>
</table>

(i) What is the optimal production schedule and the corresponding profit?  
(2 marks)

(ii) What are the optimal values of the dual variables? What do they tell us about the constraints?  
(3 marks)

(iii) Suppose the amount of raw material $M_1$ used in B can be reduced from the current 6kg/unit requirement, causing no other changes in the data in the problem. By how much does it need to be reduced to have B included into the optimal production schedule?  
(6 marks)

(iv) The manager is contemplating manufacturing another product D. A unit of D requires 4kg of $M_1$, 5kg of $M_2$, and 1.5 hours of labour. What is the minimum profit for D so that it is profitable to be included in the optimal production schedule?  
(5 marks)

(v) Denote the amount of available raw material $M_1$ by $b_1$. Find the range of $b_1$ for which the optimal production schedule remains unchanged.  
(7 marks)

(vi) Suppose additional $M_1$ can be purchased at a cost of £7.75/kg. Should more be purchased and why?  
(2 marks)
The payoff matrix for a two-person zero-sum game is given as follows:

\[ A = \begin{bmatrix} 1 & -4 \\ -3 & 4 \\ -2 & 3 \\ 0 & -5 \end{bmatrix}, \]

where the rows represent the pure strategies for player A and the columns represent those for player B.

(i) Show that the game has no pure strategy equilibrium solution.

(ii) Set up the two linear programming problems from which you can solve the optimal mixed strategy solutions for both players. Do NOT solve the resulting problem.

(iii) Find the dominated row strategy and hence reduce the payoff matrix to a $3 \times 2$ matrix.

(iv) Using the reduced payoff matrix, find the optimal strategies for the players and the value of the game (you may use graphs to assist your calculation).
(i) Given the following linear programming problem:

Maximise \( z = 5x_1 + 12x_2 + 4x_3 \)

subject to

\[
\begin{align*}
    x_1 + 2x_2 + x_3 & \leq 10 \\
    2x_1 - x_2 + 3x_3 & = 8 \\
    x_1, x_2, x_3 & \geq 0,
\end{align*}
\]

write down the canonical form of the problem, and hence find the dual problem. \((5 \text{ marks})\)

(ii) Write down the complementary slackness conditions for the primal and dual linear programming problems:

Max \( z(x) = c^T x \), \( Ax \leq b, \ x \geq 0, \)

Min \( w(y) = b^T y \), \( A^T y \geq c, \ y \geq 0. \)

Give an interpretation of the conditions in terms of shadow costs and reduced costs. \((6 \text{ marks})\)

(iii) Show that \( w(y) \geq z(x) \) for any feasible solutions \( x \) and \( y \). \((6 \text{ marks})\)

(iv) For the following linear programming problem

Maximise \( z = x_2 + 2x_3 \)

subject to

\[
\begin{align*}
    x_1 - 2x_2 + 2x_3 & \leq -8 \\
    -x_1 + x_2 + x_3 & \leq 5 \\
    2x_1 - x_2 + 4x_3 & \leq 10 \\
    x_1, x_2, x_3 & \geq 0,
\end{align*}
\]

introducing slack variables \( x_4, x_5, x_6 \geq 0 \), use the dual simplex algorithm to show that \( x_2 = 4, x_5 = 1, x_6 = 14, x_1 = x_3 = x_4 = 0 \) is a feasible basic solution (Hint: you need only perform one dual simplex step). \((8 \text{ marks})\)

End of Question Paper