1. (i) Let $X$ be a set. Prove that the set $S(X)$ of all bijections $f : X \to X$ is a group under composition of functions. 

(ii) Let $G$ be a group. For any $a \in G$ define the map $l_a : G \to G$ by the rule $l_a(x) := ax$. Prove that the map $G \to S(G) : a \mapsto l_a$ is an injective homomorphism of groups. [You may assume without proof that $l_a \in S(G)$ for all $a$.]

(iii) (a) For a group $G$ define its group of automorphisms $\text{Aut}(G)$ and the group of inner automorphisms $\text{Inn}(G)$. You should carefully define all the terms and notation used.

(b) Prove that $\text{Inn}(G)$ is a subgroup of $\text{Aut}(G)$.

(c) Determine $\text{Aut}(\mathbb{Z}/24\mathbb{Z})$ and $\text{Inn}(\mathbb{Z}/24\mathbb{Z})$ explaining your reasoning. Express $\text{Aut}(\mathbb{Z}/24\mathbb{Z})$ as a direct product of cyclic groups of prime power order.
(i) Define the centre of a group and prove that it is a normal subgroup. 

(4 marks)

(ii) (a) Let $H$ be a subgroup of $G$. Prove that $H \cap Z(G)$ is a subgroup of $Z(H)$. 

(2 marks)

(b) Give an example of $H$ and $G$ where $Z(H) \neq H \cap Z(G)$. 

(2 marks)

(iii) (a) Define the orthogonal group $O_2$ and the elements $R_\theta$, $S_\theta$ of $O_2$. 

(3 marks)

(b) By multiplying out matrices (and quoting relevant trigonometric identities) show that $R_\theta S_\phi = S_\theta + \phi$. 

(2 marks)

For the following parts you may assume that all elements of $O_2$ are of the form $R_\theta$ or $S_\theta$ for suitable $\theta$, and you may use the identities $R_\theta^{-1} = R_{-\theta}$, $S_\theta S_\phi = R_{\theta - \phi}$ and $S_\theta R_\phi = S_{\theta - \phi}$ without proving them.

(c) Determine the conjugacy class of $S_\theta$ in $O_2$. 

(2 marks)

(d) The conjugacy class of $R_\theta$ is given by $\{R_\theta, R_\theta^{-1}\}$ (no proof required for this fact). By considering the conjugacy classes of its elements determine the centre of $O_2$. 

(3 marks)

(iv) Let $T_4(Q)$ be the group of invertible $4 \times 4$ lower triangular matrices over $Q$, i.e., matrices of the form

$$
\begin{pmatrix}
    a_{11} & 0 & 0 & 0 \\
    a_{21} & a_{22} & 0 & 0 \\
    a_{31} & a_{32} & a_{33} & 0 \\
    a_{41} & a_{42} & a_{43} & a_{44}
\end{pmatrix} \in \text{GL}_4(Q).
$$

Prove that its centre is the set of scalar matrices $\{rE \mid r \in Q^*\}$, where $E$ is the $4 \times 4$ identity matrix. 

(7 marks)
(i) Give the definition of the action of a group $G$ on a set $X$.  
(3 marks)

(b) Given a homomorphism $\phi : G \to S(X)$ explain how to define an action of $G$ on $X$ and prove that it satisfies the necessary axioms.  
(4 marks)

(ii) Let $S$ be a subset of a group $G$. For $g \in G$ define 
$$g \ast S := gSg^{-1} = \{gsg^{-1} : s \in S\}.$$  
(a) Show that this defines a group action of $G$ on its set of subsets.  
(4 marks)

(b) If $H$ is a subgroup of $G$ prove that $H$ is a normal subgroup of the stabilizer of $H$ under this action, called the normalizer $N_G(H)$.  
(4 marks)

(iii) Let $C$ be a cube centred at the origin in $\mathbb{R}^3$ and write 
$$H = \text{Dir}(C) = \{A \in \text{SO}_3 | AC = C\}.$$  
(a) Describe a set of four things on which $H$ acts non-trivially, and explain carefully how this gives a homomorphism $\phi : H \to S_4$.  
(4 marks)

(b) Let $x$ be a point on the surface of $C$ that lies on an edge, close to a corner but not at the corner. Prove that the orbit of $x$ has 24 elements. By considering the orders of various sets, and assuming that $\phi$ is injective, deduce that $\phi$ is surjective.  
(6 marks)

(iv) State the Sylow theorems. You should carefully define all the terms and notation used.  
(5 marks)

(ii) Determine the number of Sylow 5-subgroups of $S_5$.  
(5 marks)

(iii) Let $G$ be a group of order 99.  
(a) Show that $G$ has a normal subgroup $N$ of order 11.  
(4 marks)

(b) Prove that if $P$ is a Sylow 3-subgroup of $G$ then every element of $P$ commutes with every element of $N$.  
(7 marks)

(c) Deduce that $G$ is abelian. [You may use without proof the fact that for every prime $p$ a group of order $p^2$ is abelian.]  
(4 marks)