1. (i) Let $S$ be a set. Give precise definitions of
(a) A $\sigma$-algebra $\Sigma$ of subsets of $S$. (3 marks)
(b) A measure on the measurable space $(S, \Sigma)$. (2 marks)

(ii) If $x \in S$, show that $\delta_x$ is a probability measure on $(S, \mathcal{P}(S))$ where for all $A \in \mathcal{P}(S)$,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

(6 marks)

(iii) Let $(S, \Sigma, m)$ be a measure space.
(a) If $A, B \in \Sigma$, use the fact that $m(A \cup B) + m(A \cap B) = m(A) + m(B)$ to deduce that $m(A \cup B) \leq m(A) + m(B)$. (1 mark)
(b) If $A_1, A_2, \ldots, A_n \in \Sigma$ use induction to prove that for all $n \geq 2$

$$m\left( \bigcup_{r=1}^{n} A_r \right) \leq \sum_{r=1}^{n} m(A_r).$$

(4 marks)
(c) If $(A_n)$ is a sequence of subsets of $S$ with $A_n \in \Sigma$ for each $n \in \mathbb{N}$, deduce that

$$m\left( \bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} m(A_n).$$

(6 marks)
(iv) Consider the subset $A = \bigcup_{n=0}^{\infty} \left( \frac{1}{22n+1}, \frac{1}{22n} \right]$ of the real line $\mathbb{R}$. Explain why this set is measurable with respect to the Borel $\sigma$-algebra of $\mathbb{R}$ and compute its Lebesgue measure. (3 marks)
Throughout this question \((S, \Sigma, m)\) is a measure space and \(\mathbb{R}\) is equipped with its usual Borel \(\sigma\)-algebra.

(i) Recall that \(f : S \to \mathbb{R}\) is a measurable function if \(f^{-1}((a, \infty)) \in \Sigma\) for all \(a \in \mathbb{R}\). Show that this is equivalent to requiring \(f^{-1}([a, \infty)) \in \Sigma\) for all \(a \in \mathbb{R}\). \hfill (4 marks)

(ii) \((a)\) Let \(f\) and \(g\) be measurable functions defined on \(S\) and define
\[
(f \wedge g)(x) = \min\{f(x), g(x)\} \text{ for all } x \in \mathbb{R}.
\]
Show that \(f \wedge g\) is measurable. \hfill (2 marks)

\((b)\) If \(f_1, f_2, \ldots, f_n\) are measurable functions on \(\mathbb{R}\), deduce that
\(f_1 \wedge f_2 \wedge \cdots \wedge f_n\) is measurable. \hfill (2 marks)

(iii) \((a)\) Suppose that \(g\) and \(h\) are measurable functions on \(S\) and \(A \in \Sigma\). For each \(x \in S\), define \(f(x) = \begin{cases} g(x) & \text{if } x \in A \\ h(x) & \text{if } x \notin A \end{cases}\).

Is \(f\) measurable? Justify your answer. \hfill (5 marks)

\((b)\) Let \((f_n)\) be a sequence of measurable functions defined on \(S\) and \((A_n)\) be a sequence of mutually disjoint sets where \(A_n \in \Sigma\) for all \(n \in \mathbb{N}\) and \(\bigcup_{n=1}^{\infty} A_n = S\). Define
\[
f(x) = f_n(x) \text{ if } x \in A_n.
\]

Is \(f\) measurable? Justify your answer. \hfill (5 marks)

(iv) Suppose that \((f_n)\) is a sequence of measurable functions defined on \(S\) and converging pointwise almost everywhere to a measurable function \(f\), so that
\[
limit_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in S - A \text{ where } A \in \Sigma \text{ with } m(A) = 0.
\]
Let \(h\) be a continuous function from \(\mathbb{R}\) to \(\mathbb{R}\). Define the functions \(G_n = h \circ f_n\) for each \(n \in \mathbb{N}\) and \(G = h \circ f\).

\((a)\) Explain why \(G\) and \(G_n\) (for all \(n \in \mathbb{N}\)) are measurable. \hfill (2 marks)

\((b)\) Deduce that the sequence \((G_n)\) converges pointwise almost everywhere to \(G\). \hfill (5 marks)
Throughout this question \((S, \Sigma, m)\) is a measure space and \(\mathbb{R}\) is equipped with its usual Borel \(\sigma\)-algebra.

(i) (a) Explain how to define \(\int_S f \, dm\) in the case where \(f : S \rightarrow \mathbb{R}\) is a non-negative simple function, i.e. \(f = \sum_{i=1}^{n} c_i 1_{A_i}\) where \(c_1, \ldots, c_n \in [0, \infty)\) and \(A_1, \ldots, A_n \in \Sigma\) for some \(n \in \mathbb{N}\) with \(\bigcup_{n=1}^{\infty} A_n = S\) and \(A_i \cap A_j = \emptyset\) when \(i \neq j\). \(2\) marks

(b) Explain how to extend the definition of \(\int_S f \, dm\) to the case where \(f : S \rightarrow \mathbb{R}\) is an arbitrary non-negative measurable function. What does it mean for such an \(f\) to be integrable? \(3\) marks

(ii) Suppose that \(f\) and \(g\) are non-negative simple functions defined on \(S\). Show that

\[
\int_S (f + g) \, dm = \int_S f \, dm + \int_S g \, dm.
\]

Extend this result to the case where \(f\) and \(g\) are arbitrary non-negative measurable functions, stating carefully any results that you use to deduce this. \(12\) marks

(iii) (a) Let \(a \in \mathbb{R}\). Explain why the mapping \(x \rightarrow \frac{1}{a^2 + x^2}\) is integrable with respect to Lebesgue measure on \([0, \infty)\). \(5\) marks

(b) Deduce that the mapping \(x \rightarrow \frac{e^{-bx}}{a^2 + x^2}\) is integrable with respect to Lebesgue measure on \([0, \infty)\), where \(b > 0\). \(3\) marks
Throughout this question \((S, \Sigma, m)\) is a measure space and \(\mathbb{R}\) is equipped with its usual Borel \(\sigma\)-algebra.

(i) State the monotone convergence theorem and use it to prove Fatou’s lemma, i.e. if \((f_n)\) is a sequence of non-negative functions from \(S\) to \(\mathbb{R}\) then

\[
\liminf_{n \to \infty} \int_S f_n \,dm \geq \int_S \liminf_{n \to \infty} f_n \,dm.
\]

\(8 \text{ marks}\)

(ii) (a) Deduce the reverse Fatou lemma, i.e. if \((f_n)\) is a sequence of non-negative measurable functions for which \(f_n \leq f\) for all \(n \in \mathbb{N}\) where \(f\) is integrable then

\[
\limsup_{n \to \infty} \int_S f_n \,dm \leq \int_S \limsup_{n \to \infty} f_n \,dm.
\]

\(5 \text{ marks}\)

[b]Hint. Apply Fatou’s lemma to \(f - f_n\).[/b]

(b) Show that the reverse Fatou lemma fails to work in the case where \(S\) is the real number line equipped with Lebesgue measure and \(f_n = 1_{(n, n+1]}\) for each \(n \in \mathbb{N}\) and comment on why there is no contradiction here with the result just proved. \(3 \text{ marks}\)

(iii) (a) Let \((S, \Sigma, m)\) be a measure space and \(f : [a, b] \times S \to \mathbb{R}\) be a measurable function for which

(I) \ The mapping \(x \to f(t, x)\) is integrable for all \(t \in [a, b]\),

(II) \ The mapping \(t \to f(t, x)\) is continuous for all \(x \in S\),

(III) \ There exists a non-negative integrable function \(g : S \to \mathbb{R}\) so that \(|f(t, x)| \leq g(x)\) for all \(t \in [a, b], x \in S\).

Use the dominated convergence theorem to show that the mapping \(t \to \int_S f(t, x) \,dm(x)\) is continuous on \([a, b]\). \(6 \text{ marks}\)

(b) If \(f : \mathbb{R} \to \mathbb{R}\) is integrable, deduce that the mapping from \([0, 1]\) to \(\mathbb{R}\) given by \(t \to \int_\mathbb{R} f(x)/(1+t) \,dx\) is continuous. \(3 \text{ marks}\)
Throughout this question all random variables are defined on a common probability space \((\Omega, \mathcal{F}, P)\).

(i)  (a) Write the expectation \(\mathbb{E}(X)\) of an integrable random variable \(X\) as a Lebesgue integral with respect to the measure \(P\) and then use expectation to define the variance \(\text{Var}(X)\).  

\[ (2 \text{ marks}) \]

(b) If \(X\) and \(Y\) are integrable random variables such that \(X(\omega) \leq Y(\omega)\) for all \(\omega \in \Omega\), explain briefly why integration theory enables us to conclude that \(\mathbb{E}(X) \leq \mathbb{E}(Y)\).  

\[ (2 \text{ marks}) \]

(ii)  (a) Suppose that \(X\) and \(Y\) are random variables wherein both \(X^2\) and \(Y^2\) are integrable. Prove the Cauchy-Schwarz inequality:

\[ |\mathbb{E}(XY)| \leq (\mathbb{E}(X^2)^{1/2})(\mathbb{E}(Y^2)^{1/2}). \]

\[ (4 \text{ marks}) \]

[Hint: Consider \(g(t) = \mathbb{E}((X + tY)^2)\) as a quadratic function of \(t \in \mathbb{R}\).]

(b) Deduce that if \(X^2\) is integrable, then so is \(X\) and that

\[ |\mathbb{E}(X)|^2 \leq \mathbb{E}(X^2). \]

\[ (3 \text{ marks}) \]

(iii) Deduce that for a random variable \(X\) having a finite mean \(\mu\), \(\mathbb{E}(X^2) < \infty\) if and only if \(\text{Var}(X) < \infty\). Show further that \(\mathbb{E}(|X - \mu|^2) \leq \text{Var}(X)\).

\[ (4 \text{ marks}) \]

(iv) Let \(X\) be a random variable for which \(\mathbb{E}(e^{t|X|}) < \infty\) for all \(t > 0\).

(a) Deduce that \(\mathbb{E}(e^{tX}) < \infty\) for all \(t \in \mathbb{R}\).  

\[ (2 \text{ marks}) \]

(b) Show that \(\mathbb{E}(|X^n|) < \infty\) for all \(n \in \mathbb{N}\).  

\[ (2 \text{ marks}) \]

(c) Prove that \(\mathbb{E}(X) = \frac{d}{dt}\mathbb{E}(e^{tX})\bigg|_{t=0} \), giving careful details of the use of appropriate convergence theorems. [Hint: Use the mean value theorem.]  

\[ (6 \text{ marks}) \]

End of Question Paper