Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

Throughout the paper I denotes an identity matrix and J denotes a matrix of the form\[
\begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}.
\]
All matrices have real entries. The standard symplectic form $\Omega$ on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$. 

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to be completed by student
1. (i) Define what it means for \((V, \omega)\) to be a \textit{symplectic vector space}.

Define what it means for a vector subspace \(W \subseteq V\) to be a \textit{symplectic subspace} of \((V, \omega)\).

Define the \textit{symplectic perp} or \textit{skew} \(W^\wedge\) of \(W\). \hspace{3cm} (5 marks)

(ii) Let \((V, \omega)\) be a symplectic vector space of dimension \(2n \geq 2\). Prove that there is a subspace \(W \subseteq V\) of dimension 2 such that \(W^\wedge\) is symplectic and \(V = W \oplus W^\wedge\). \hspace{3cm} (12 marks)

(iii) Let \((V, \omega)\) be a symplectic vector space of dimension \(2n\) and let \(W \subseteq V\) be a vector subspace. Using the fact that \(\dim W^\wedge = 2n - \dim W\) or otherwise, prove that \((W^\wedge)^\wedge = W\). \hspace{3cm} (3 marks)

(iv) Let \((V, \omega)\) be a symplectic vector space and let \(W \subseteq V\) be a vector subspace. Show that \(W\) is a symplectic subspace if and only if \(W \cap W^\wedge = \{0\}\). \hspace{3cm} (2 marks)

(v) Let \((V, \omega)\) be a symplectic vector space and let \(W \subseteq V\) be a vector subspace. Is it true that if \(W\) is a symplectic subspace, then \(W^\wedge\) is also? Either prove this, or exhibit a counterexample. \hspace{3cm} (3 marks)

2. Let \(\mathcal{L}_2\) denote the set of all oriented straight lines in the plane \(\mathbb{R}^2\).

(a) Define, using sketch diagrams if you wish, a bijective map from \(\mathcal{L}_2\) to \(U \times \mathbb{R}\), where \(U\) is a unit circle in the plane.

Describe carefully the map \(U \times \mathbb{R} \to \mathcal{L}_2\) which is inverse to your map. \hspace{3cm} (6 marks)

(b) Describe carefully a bijective map \(\mathcal{L}_2 \to M\) where

\[
M := \{(X, Y) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid |X|^2 = 1, \ X \cdot Y = 0\}
\]

and its inverse map \(M \to \mathcal{L}_2\). \hspace{3cm} (3 marks)

(c) Fix a point \((X, Y) \in M\) and consider a smooth curve in \(M\) given by \((X(t), Y(t))\) for \(t \in \mathbb{R}\) with \(X(0) = X,\ Y(0) = Y\).

Writing \(Q = \dot{X}(0),\ P = \dot{Y}(0)\), show that

\[
Q \cdot X = 0, \quad Q \cdot Y + X \cdot P = 0. \quad (*)
\]

Conversely show that every \((Q, P) \in \mathbb{R}^2 \times \mathbb{R}^2\) which satisfies the conditions \((*)\) is the derivative of a curve at \((X, Y)\) \hspace{3cm} (10 marks)

(d) Writing

\[
T_{(X,Y)} = \{(Q, P) \mid Q \cdot X = 0, \ Q \cdot Y + X \cdot P = 0\},
\]

show that \(T_{(X,Y)}\) is a symplectic subspace of \(\mathbb{R}^2 \times \mathbb{R}^2\) with respect to \(\Omega\). \hspace{3cm} (6 marks)
Figure 1: For Question 3(i). The central vertical line represents a plane in $\mathbb{R}^3$.

(i) Figure 1 shows two unit vectors $v, v'$ in $\mathbb{R}^3$ and a third unit vector $\Sigma$ which is normal to the vertical plane separating the region of index of refraction $n$ from the region of index of refraction $n'$.

Starting from Snell’s Law in the form $n \sin \theta = n' \sin \theta'$, and referring to the situation in Figure 1, obtain a vector equation expressing Snell’s Law in terms of $v, v', n, n'$ and $\Sigma$. (8 marks)

(ii) Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a $2n \times 2n$ matrix in block form, where $A, B, C$ and $D$ denote $n \times n$ matrices.

(a) Prove that $S$ is symplectic if and only if the three equations

$$A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I,$$

hold.

(b) Let $W$ be an $n$-dimensional subspace of $\mathbb{R}^{2n}$, and let $G_1, \ldots, G_n$ be a basis for $W$. Write the matrix which has $G_1, \ldots, G_n$ as its columns in block form as $[M_N]^T$.

Show that $W$ is Lagrangian with respect to $\Omega$ if and only if $M^T N$ is symmetric.

(c) Let $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be a $2n \times 2n$ symplectic matrix. Find a simple condition on $B$ or on $D$ for the Lagrangian subspace corresponding to $[B \ D]$ to be transversal to $0 \times \mathbb{R}^n$. (13 marks)

(iii) Let $(V_1, \omega_1)$ and $(V_2, \omega_2)$ be symplectic vector spaces. A symplectic map $\varphi: V_1 \to V_2$ is a linear map such that $\omega_2(\varphi(v), \varphi(v')) = \omega_1(v, v')$ for all $v, v' \in V_1$.

Show that a symplectic map $\varphi: V_1 \to V_2$ is injective. (4 marks)
(i) In linear optics the matrix which transforms variables \((Q, P)\) at \(z\) to variables \((Q', P')\) at \(z'\) is \[
\begin{bmatrix}
I & w \\
0 & I
\end{bmatrix}
\] where \(w = \frac{z' - z}{n}\) and \(n\) is the index of refraction.

Further, given a paraboloid boundary \(z = z_0 + \frac{1}{2}(f q_1^2 + 2gq_1q_2 + hq_2^2)\) between media with indexes of refraction \(n\) and \(n'\), the matrix which transforms variables \((Q, P)\) to variables \((Q', P')\) is \[
\begin{bmatrix}
I & 0 \\
M & I
\end{bmatrix}
\] where \(M = (n' - n) \begin{bmatrix} f & g \\ g & h \end{bmatrix}\).

(a) Use these facts to find the matrix which describes the following situation: two media of refractive indexes \(n\) (to the left) and \(n'\) (to the right) are separated by a surface given by

\[
z = z_0 + \frac{1}{2}(f q_1^2 + 2gq_1q_2 + hq_2^2),
\]

the variables \((Q, P)\) are at \(z = z_0 - k\), and the variables \((Q', P')\) are at \(z = z_0 + k'\) where \(k, k' > 0\). (5 marks)

(b) Consider rays which are parallel to the \(z\)-axis at \(z = z_0 - k\), and write \(w' = \frac{k'}{n'}\).

In each case below describe in a few words the light which emerges at \(z = z_0 + k'\):

- \(-\frac{1}{w'}\) is not an eigenvalue of \(M\);
- \(-\frac{1}{w'}\) is an eigenvalue of \(M\) and the other eigenvalue is distinct;
- \(-\frac{1}{w'}\) is a double eigenvalue of \(M\). (8 marks)

(ii) Let \(M\) be an \(n \times n\) matrix, and write

\[
W = \{(Q, P) \mid P = MQ\}.
\]

When is \(L\) Lagrangian in \(\mathbb{R}^{2n}\) with respect to \(\Omega\)? (6 marks)

(iii) Let \(A\) be a symmetric \(n \times n\) matrix, let \(C\) be a symmetric \(k \times k\) matrix, and let \(B\) be any \(n \times k\) matrix. Prove that \(\Omega\) is zero on

\[
L = \{(Q, AQ + BX) \mid Q \in \mathbb{R}^n, X \in \mathbb{R}^k, B^TQ + CX = 0\};
\]

that is, show that \(\Omega(Z, Z') = 0\) for \(Z, Z' \in L\). (6 marks)
Take $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in Sp(2)$ with $b \neq 0$. Show that a choice of $q$ and $q'$ determine $p$ and $p'$ such that
\[
\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}.
\]

Define $\Gamma: \mathbb{R}^2 \to \mathbb{R}$ by $\Gamma(q, q') = \frac{1}{5} (qq' - \frac{1}{2}aq^2 - \frac{1}{2}dq'^2)$.

Show that
\[
\frac{\partial \Gamma}{\partial q} = p, \quad \frac{\partial \Gamma}{\partial q'} = -p'.
\]

(ii) Define when a smooth map $\varphi: \mathbb{R}^4 \to \mathbb{R}^4$ is a symplectomorphism.

Writing $\varphi(Q, P) = (F(Q, P), G(Q, P))$, formulate your definition in terms of the partial derivative matrices of $\varphi$. You may use the result of Question 3(ii)(a) without proof.

(iii) Let $\varphi: \mathbb{R}^4 \to \mathbb{R}^4$ be a symplectomorphism and write $\varphi(Q, P) = (F(Q, P), G(Q, P))$ as above. Assume that $\det \left( \frac{\partial F}{\partial P} \right) \neq 0$.

(a) The Local Diffeomorphism Theorem implies that it is possible to solve the equation $Q' = F(Q, P)$ for $P$ in terms of $Q$ and $Q'$. That is, there is a smooth function $H(Q, Q')$, defined in some open set, such that
\[
Q' = F(Q, P) \text{ if and only if } P = H(Q, Q'). \tag{*}
\]

Assuming this, show that
\[
\frac{\partial F}{\partial Q} + \frac{\partial F}{\partial P} \frac{\partial H}{\partial Q} = 0, \quad \frac{\partial F}{\partial P} \frac{\partial H}{\partial Q'} = I. \tag{**}
\]

(b) Now define $K(Q, Q') = -G(Q, H(Q, Q'))$. It follows that
\[
\frac{\partial K}{\partial Q} = \frac{\partial G}{\partial Q} - \frac{\partial G}{\partial P} \frac{\partial H}{\partial Q}, \quad \frac{\partial K}{\partial Q'} = -\frac{\partial G}{\partial P} \frac{\partial H}{\partial Q'}. \tag{***}
\]

Using these, show that
\[
\left( \frac{\partial K}{\partial Q} \right)^T = \frac{\partial H}{\partial Q'}.
\]

(16 marks)

End of Question Paper