SCHOOL OF MATHEMATICS AND STATISTICS  Spring Semester 2013–2014

Algebra I  2 hours 30 minutes

Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

Please leave this exam paper on your desk
Do not remove it from the hall
Registration number from U-Card (9 digits)
to be completed by student
1. (i) Consider the cubic equation

\[ t^3 + pt + q = 0, \]  \hfill (∗)

where \( p \) and \( q \) are real numbers.

(a) Show that if \( (∗) \) has repeated roots then \( 4p^3 + 27q^2 = 0 \).

(b) Show that if \( (∗) \) has one real and two (non-real) complex roots, then \( 4p^3 + 27q^2 > 0 \). \hfill (9 marks)

(ii) Define the following concepts:

- the characteristic of a field,
- a homomorphism of fields,
- the degree of a homomorphism of fields,
- an automorphism of a field,
- an ideal in a ring. \hfill (9 marks)

(iii) Suppose that \( ϕ: K → L \) is a homomorphism of fields.

(a) Show that \( ϕ \) is injective.

(b) Show that \( K \) and \( L \) have the same characteristic. \hfill (7 marks)

2. A polynomial \( f(x) = \sum_{i=0}^{d} a_i x^i \in \mathbb{Z}[x] \) is primitive if the greatest common divisor of \( a_0, \ldots, a_d \) is 1.

(a) Prove that if \( f(x) \) and \( g(x) \) are primitive polynomials in \( \mathbb{Z}[x] \), then so is \( f(x)g(x) \). \hfill (5 marks)

(b) Let \( q(x) \) be a monic polynomial in \( \mathbb{Z}[x] \), and suppose that there is a factorisation \( q(x) = f(x)g(x) \) with both of \( f(x) \) and \( g(x) \) monic polynomials in \( \mathbb{Q}[x] \).

Show that in fact \( f(x) \) and \( g(x) \) lie in \( \mathbb{Z}[x] \). \hfill (7 marks)

(c) List the quadratic polynomials over \( \mathbb{F}_2 \) and establish whether they are reducible or irreducible. \hfill (4 marks)

(d) Show that the polynomial \( x^5 + x^2 + 1 \) is irreducible in \( \mathbb{F}_2[x] \).

Deduce, using (b) and (c) or otherwise, that the polynomial \( x^5 + x^2 + 1 \) is also irreducible in \( \mathbb{Q}[x] \). \hfill (9 marks)
(a) Let $L$ and $M$ be fields, and let $\theta_1, \ldots, \theta_n : L \to M$ be $n$ distinct homomorphisms. Let $b_1, \ldots, b_n \in M$ and suppose that for all $a \in L$ we have

$$\sum_{i=1}^{n} b_i \theta_i(a) = 0.$$ 

Show that $b_1 = b_2 = \cdots = b_n = 0$. (10 marks)

(b) Now let $K$ be another field and let $\phi : K \to L$ and $\psi : K \to M$ be field homomorphisms with $\deg(\phi) < \infty$.

Write $E(\phi, \psi)$ for the set of homomorphisms $\theta : L \to M$ with $\theta \phi = \psi$.

Using (a) or otherwise, show that $|E(\phi, \psi)| \leq \deg(\phi)$. (10 marks)

(c) Let $N/K$ be a field extension of finite degree. Explain what it means for $N$ to be normal over $K$. Give one criterion in terms of roots of polynomials, and another criterion in terms of numbers of homomorphisms. (5 marks)

Put $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{7})$.

(a) Write down a basis for $L$ over $\mathbb{Q}$. (You are not asked to prove that your answer is correct.) (3 marks)

In the rest of the question you may assume without proof that $L/\mathbb{Q}$ is Galois.

(b) List without proof the elements of the group $G(L/\mathbb{Q})$. To which well-known group is $G(L/\mathbb{Q})$ isomorphic? (6 marks)

(c) For each of the following fields $K_i$, determine the subgroup $H_i \leq G(L/\mathbb{Q})$ that corresponds to $K_i$ under the Galois correspondence.

$K_1 = \mathbb{Q}(\sqrt{14})$, $K_2 = \mathbb{Q}(\sqrt{6}, \sqrt{21})$, $K_3 = \mathbb{Q}(\sqrt{2} + \sqrt{7})$, $K_4 = \mathbb{Q}(\sqrt{42})$. (7 marks)

(d) Use the Galois correspondence to show that $K_1 \leq K_3$, and then prove the same thing by a direct calculation. (4 marks)

(e) How many fields $M$ are there with $\mathbb{Q} < M < L$ and $[M : \mathbb{Q}] = 4$? (5 marks)

Consider the polynomial $f(x) = x^4 + 8x^2 - 2 \in \mathbb{Q}[x]$.

Define $\alpha = \sqrt[3]{3\sqrt{2} - 4}$, and $M = \mathbb{Q}(\alpha, \sqrt{-2})$.

(a) Show that $f(x)$ is irreducible over $\mathbb{Q}$, stating clearly, without proof, any general criterion which you use. (5 marks)

(b) Show that $f(x)$ has roots $\pm \alpha, \pm \sqrt{-2}/\alpha$. Deduce that $M$ is a splitting field for $f(x)$. (7 marks)

(c) Show that $\mathbb{Q}(\alpha) = M \cap \mathbb{R} \neq M$, and deduce that $[M : \mathbb{Q}] = 8$. (5 marks)

(d) Show that there exist automorphisms $\varphi, \psi \in G(M/\mathbb{Q})$ such that $\varphi$ has order 4, $\psi$ has order 2, and $G(M/\mathbb{Q}) = \langle \varphi, \psi \rangle$. (8 marks)