SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2013–2014

MAS6450 Waves and Magnetohydrodynamics 2 hours

Answer all four questions.

1 (i) Consider a magnetic field with components

\[ B_x(x, z) = \frac{\partial \psi}{\partial z}, \quad B_y = B_y(x, z), \quad B_z(x, z) = -\frac{\partial \psi}{\partial x}, \]

where \( \psi = \psi(x, z) \).

(a) Show that \( \nabla \cdot \mathbf{B} = 0 \). (2 marks)

(b) Show that \( \mathbf{B} \cdot \nabla \psi = 0 \) and that projections of field lines in the \( xz \)-plane are given by \( \psi = \text{constant} \). (6 marks)

(c) Show that if \( \mathbf{J} \times \mathbf{B} = 0 \), where \( \mathbf{J} \) is the current density, then

\[ B_y = B_y(\psi) \]

and \( \psi \) satisfies

\[ \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial z^2} + B_y \frac{dB_y}{d\psi} = 0. \] (12 marks)

(ii) Calculate the approximate timescale (in years) for the decay of the interstellar magnetic field given the parameters \( L = 3 \times 10^{16} \text{ m} \) and \( \eta = 3.6 \times 10^6 \text{ cm}^2 \text{ s}^{-1} \). (5 marks)

2 (i) State the relationship between Lagrangian and Eulerian perturbations. Write the Lagrangian density perturbation in terms of the Eulerian perturbation. (4 marks)

Question 2 continued on next page
(ii) Consider
\[ \mathbf{B} = B_0(y/a, x/a, 0). \]
Sketch the field lines showing the direction. \(4 \text{ marks}\)

Calculate the magnetic tension and magnetic pressure forces. Comment on the direction of each. Give comments about the Lorentz force. \(7 \text{ marks}\)

(iii) Prove that
\[ \frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}, \]
where \(r\) and \(\theta\) are two-dimensional polar coordinates with \(\hat{r}\) and \(\hat{\theta}\) being the corresponding unit vectors. \(3 \text{ marks}\)

(iv) Prove, using (iii), that for magnetic fields of the form \(\mathbf{B} = B_\theta(r)\hat{\theta}\),
\[ (\mathbf{B} \cdot \nabla)\mathbf{B} = -\frac{B_\theta^2}{r}\hat{r}. \]
\(3 \text{ marks}\)

(i) (a) Ignoring viscosity, gravity and diffusivity, derive the linearised induction equation and equation of motion for adiabatic, ideal perturbation of the form \(\sim e^{ikr-\omega t}\) about a static and uniform equilibrium state with magnetic force alone. \(6 \text{ marks}\)

[You may use \(\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} - (\nabla \cdot \mathbf{B})\mathbf{A} + (\nabla \cdot \mathbf{A})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}\).]

(b) Derive the dispersion relations for Alfvén and compressional Alfvén waves using linearised MHD equations obtained in (i). \(8 \text{ marks}\)

(c) State one property of each:
(a) Alfvén waves
(b) compressional Alfvén waves. \(2 \text{ marks}\)

(ii) (a) Write the induction equation for a perfectly conducting medium. Given a velocity field \(\mathbf{v} = (yz, -xz, 0)\) and the initial condition on the magnetic field
\[ \mathbf{B}(x, 0) = (x, -y, 0), \]
find \(\mathbf{B}(x, t)\) by obtaining the Lagrangian coordinates corresponding to \(\mathbf{v}\) and applying the Cauchy solution. \(19 \text{ marks}\)
Hint: Use an initial condition $x(0) = (a_1, a_2, a_3)$ and also make use of $x(0)$ for the initial magnetic field $B(x, 0) = B_0$.

(b) Having obtained $B(x, t)$ in (iv), verify by direct substitution that it is indeed the solution of the induction equation. \(4 \text{ marks}\)

4 (i) Consider a cartesian coordinate system where the magnetic and velocity fields are independent of $y$, i.e.

\[
B = \begin{pmatrix} -\frac{\partial A}{\partial z}, B, \frac{\partial A}{\partial x} \end{pmatrix},
\]

\[
v = \begin{pmatrix} -\frac{\partial \psi}{\partial z}, v_y, \frac{\partial \psi}{\partial x} \end{pmatrix},
\]

where $A$ and $\psi$ are scalar potential functions for magnetic and flow fields, respectively. Using the magnetic induction equation, the equations for the evolution of $B$ and $A$ can be given as

\[
\frac{\partial B}{\partial t} + \left( \frac{\partial \psi}{\partial x} \frac{\partial B}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial B}{\partial x} \right) = \left( \frac{\partial A}{\partial x} \frac{\partial v_y}{\partial z} - \frac{\partial A}{\partial z} \frac{\partial v_y}{\partial x} \right) - \left( \nabla \cdot (\alpha \nabla A) - \eta \nabla^2 B \right), \tag{I}
\]

\[
\frac{\partial A}{\partial t} + \left( \frac{\partial \psi}{\partial x} \frac{\partial A}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial A}{\partial x} \right) = \alpha B + \eta \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] A. \tag{II}
\]

Simplify the above equations (I and II) by setting $\psi = 0$, $v_y = Qz$ and ignoring the term containing $\alpha$ in the equation for the evolution of $B$.

\(2 \text{ marks}\)

(ii) Assuming solutions of the form $(A, B) = (\tilde{A}, \tilde{B})e^{\sigma t - i(k_x x + k_z z)}$ in the simplified equations obtained in (i), show that the following condition is satisfied for the solutions to exist

\[(\sigma + \eta \kappa^2)^2 = -ik_z \alpha Q,\]

where $\kappa^2 = k_x^2 + k_z^2$. \(6 \text{ marks}\)

(iii) In the equations for the evolution of $B$ and $A$ in (i), assume $\psi = v_y = 0$ but retain the term containing $\alpha$ to be a constant. Substitute the same solutions of the form $(A, B) = (\tilde{A}, \tilde{B})e^{\sigma t - i(k_x x + k_z z)}$ and show that now the solution satisfies the relation

\[\sigma = \pm \alpha \kappa - \eta \kappa^2.\]

\(7 \text{ marks}\)

End of Question Paper