Section A

A1 Find the general solution of the equation
\[ x^2 \frac{dy}{dx} - y = \frac{e^{-1/x}}{x} \] (7 marks)

A2 Find the particular solution of the equation
\[ \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = 2e^{-3x} \]
which satisfies \( \frac{dy}{dx} = 0 \) and \( y = 1 \) when \( x = 0 \). (13 marks)

A3 The pressure \( (p) \), volume \( (V) \) and temperature \( (T) \) of a container full of gas are related by the ideal gas law
\[ pV = nRT, \]
where \( n \) measures the number of gas particles and \( R \) is the universal gas constant. The pressure is initially \( 10^5 \) Nm\(^{-2}\). Using the chain rule, estimate the pressure if the volume decreases by 4\% and the temperature increases by 3\%. (7 marks)
A4 Two quantities $x$ and $y$ have means 2.36 and 73.4 respectively, standard deviations 3.18 and 2.19 respectively, and covariance -6.87.

(a) Calculate the correlation coefficient between $x$ and $y$, correct to 3 significant figures. (2 marks)

(b) It is assumed that $x$ and $y$ satisfy the linear relationship

$$y = a + b(x - \bar{x}),$$

where $\bar{x}$ is the mean of $x$.

Calculate the least squares estimates of $a$ and $b$, correct to 3 significant figures. State, giving a reason, whether you expect (*) to give a good model. (4 marks)

A5 Find a vector normal to the surface $\phi = 5$ at the point $A$ with coordinates $(1, 2, 3)$, where

$$\phi = x^2 \cos \pi z + xyz - z \sin \pi y.$$ (5 marks)

Find also the directional derivative of $\phi$ at $A$, in the direction $d = (2, 0, -1)$. (2 marks)

Section B

B1 (a) For $x > 0$, find the particular solution of the equation

$$x \frac{dy}{dx} = y + \frac{x^3}{y^2}$$

which satisfies $y = 3$ when $x = 1$. (11 marks)

(b) Find the general solution of the equation

$$4 \frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 5y = x.$$ (9 marks)

B2 (a) Let $R = (x^2 + y^2)^{1/2}$, and define a scalar field $\phi$ by

$$\phi = \ln R.$$

Find $\nabla \phi$, and show that

$$\nabla^2 \phi = 0.$$ (12 marks)

(b) A vector field $u$ is given by

$$u = (x + y \sin z, y + e^z, xz - \cosh y).$$

Verify that

$$\nabla \cdot (\nabla \times u) = 0,$$

and find $\nabla \cdot u$. (8 marks)
B3 A function $f(x) = x^2$ is defined on the interval $0 \leq x \leq 1$.
(a) Show that $f(x)$ can be represented by the Fourier series

$$\frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(n\pi x).$$

[You may use the fact that $\cos(n\pi) = (-1)^n$.] \hspace{1cm} (15 marks)

(b) Use the result of part (a) to find

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$ \hspace{1cm} (5 marks)

B4 The function $\phi(x, y)$ satisfies Laplace’s equation in two dimensions, i.e.

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0,$$

in the square region $0 < x < 1$, $0 < y < 1$. The boundary conditions are

$$\begin{align*}
\phi(0, y) &= 0 \\
\phi(x, 0) &= 0 \\
\phi(x, 1) &= 0 \\
\phi(1, y) &= 1.
\end{align*}$$

If $\phi(x, y) = X(x)Y(y)$ for some functions $X$ and $Y$, show that

$$\frac{X''}{X} = \frac{Y''}{Y} = \alpha,$$

where $\alpha$ must be a constant. \hspace{1cm} (3 marks)

Find the values of $X(0)$, $Y(0)$ and $Y(1)$. \hspace{1cm} (3 marks)

Given that $\alpha > 0$, deduce that

$$\phi(x, y) = \sum_{n=1}^{\infty} A_n \sinh n\pi x \sin n\pi y$$

for some constants $A_n$. \hspace{1cm} (8 marks)

Show that

$$A_n = \begin{cases} 
0 & \text{n even} \\
\frac{4}{n\pi \sinh n\pi} & \text{n odd.}
\end{cases}$$ \hspace{1cm} (6 marks)

End of Question Paper
FORMULA SHEET

Trigonometry

\[ 1 + \tan^2 \theta = \sec^2 \theta \]
\[ 1 + \cot^2 \theta = \cosec^2 \theta \]
\[ \cos(A + B) = \cos A \cos B - \sin A \sin B \]
\[ \cos(A - B) = \cos A \cos B + \sin A \sin B \]
\[ \sin(A + B) = \sin A \cos B + \cos A \sin B \]
\[ \sin(A - B) = \sin A \cos B - \cos A \sin B \]
\[ \tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \]
\[ \tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B} \]
\[ \sin 2\theta = 2 \sin \theta \cos \theta \]
\[ \cos 2\theta = 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \]
\[ a \cos \theta + b \sin \theta = R \cos(\theta - \alpha), \text{ where } R = \sqrt{(a^2 + b^2)}, \cos \alpha = a/R \text{ and } \sin \alpha = b/R \]

Hyperbolic Functions

\[ \sinh x = \frac{1}{2}(e^x - e^{-x}) \]
\[ \cosh x = \frac{1}{2}(e^x + e^{-x}) \]
\[ \cosh^2 x - \sinh^2 x = 1 \]
\[ \text{sech}^2 x + \tanh^2 x = 1 \]
\[ 2 \sinh x \cosh x = \sinh 2x \]
\[ \cosh 2x = 2 \cosh^2 x - 1 = 2 \sinh^2 x + 1 \]
\[ \sinh^{-1} x = \ln \left[ x + \sqrt{(1 + x^2)} \right], \quad \text{all } x \]
\[ \cosh^{-1} x = \ln \left[ x + \sqrt{(x^2 - 1)} \right], \quad x \geq 1 \]
\[ \tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1 + x}{1 - x} \right), \quad |x| < 1 \]
\[ \coth^{-1} x = \frac{1}{2} \ln \left( \frac{x + 1}{x - 1} \right), \quad |x| > 1 \]
<table>
<thead>
<tr>
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<th>Derivative</th>
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<tr>
<td>$x^n$</td>
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<td>$\cosh^{-1} \left( \frac{x}{a} \right)$</td>
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Differentiation and Integration Formulae

\[
\frac{d(uv)}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}
\]

\[
\frac{d}{dx} \left( \frac{u}{v} \right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
\]

\[
\int_a^b uvdx = \left[ u \times (\text{integral of } v) \right]_a^b - \int_a^b \frac{du}{dx} \times (\text{integral of } v)dx
\]

Partial Differentiation

Chain Rule

1. Suppose that $z = f(x, y)$ and that $x$ and $y$ are functions of $t$, i.e., $x = x(t)$, $y = y(t)$. Then

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

2. Suppose that $z = f(x, y)$ and that $x$ and $y$ are functions of the variables $r$ and $s$, i.e., $x = x(r, s)$, $y = y(r, s)$. Then

\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}, \quad \frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s}
\]
First-Order Differential Equations

1. Direct Integration

\[ \frac{dy}{dx} = f(x) \]

\[ y = \int f(x) dx + C \]

2. Separation of Variables

\[ \frac{dy}{dx} = f(x) g(y) \]

\[ \int \frac{dy}{g(y)} = \int f(x) dx \]

3. Homogeneous Equations

\[ \frac{dy}{dx} = f \left( \frac{y}{x} \right) \]

make the substitution \( y = zx \) to give

\[ z + x \frac{dz}{dx} = f(z) \]

4. Linear Equations

\[ \frac{dy}{dx} + P(x) y = Q(x) \]

multiply both sides by the integrating factor \( e^{\int P(x) dx} \) to give

\[ \frac{d}{dx} \left( ye^{\int P(x) dx} \right) = Q(x) e^{\int P(x) dx} \]
The Second-Order Differential Equation

\[ a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x) \]

where \( a, b, \) and \( c \) are constants.

General solution is

\[ y = \text{Complementary Function} + \text{Particular Integral} \]

The solution, \( y_c, \) is given by

(i) \( y_c = Ae^{m_1x} + Be^{m_2x}, \) if \( m_1 \) and \( m_2 \) real and different,

(ii) \( y_c = e^{mx}(A + Bx), \) if \( m_1 \) and \( m_2 \) real and equal \( (m_1 = m_2 = m), \)

(iii) \( y_c = e^{px}(A \cos mx + B \sin mx), \) if \( m_1 \) and \( m_2 \) are complex \( (m_1 = p + iq, m_2 = p - iq), \)

where \( m_1 \) and \( m_2 \) are the roots of the auxiliary equation

\[ am^2 + bm + c = 0 \]

Particular Integral, \( y_p \)

\[ f(x) = Ax^2 + Bx + C \quad y_p = ax^2 + bx + c \]

\[ f(x) = Ae^{kx} \quad y_p = ae^{kx} \]

when \( k \) is not one of the roots of the auxiliary equation

\[ f(x) = Ae^{kx} \quad y_p = axe^{kx} \]

when \( k \) is one of the roots of the auxiliary equation

\[ f(x) = A \cos mx + B \sin mx \quad y_p = a \cos mx + b \sin mx \]

when \( \sin mx \) or \( \cos mx \) is not part of the complementary function

\[ f(x) = A \cos mx + B \sin mx \quad y_p = x(a \cos mx + b \sin mx) \]

when \( \sin mx \) or \( \cos mx \) is part of the complementary function
Fourier Series
Suppose that $f(x)$ is defined on the interval $-l \leq x \leq l$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right),$$

where

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} \, dx, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx, \quad n = 0, 1, 2, \ldots.$$

On the interval $0 \leq x \leq l$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad a_n = \frac{2}{l} \int_{0}^{l} f(x) \cos \frac{n\pi x}{l} \, dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \quad b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

Vector Calculus
The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla \phi = \left( \frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right).$$

The divergence of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

The curl of a vector field $\mathbf{u}(x, y, z) = (u, v, w)$ is given by

$$\nabla \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

The Laplacian $\nabla^2$ is given by

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$
Statistics

For data values \((x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)\)

Means \(\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i\) etc.

Variances \(s_x^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2\) etc.

\(s_x\) is standard deviation

Covariance \(\text{cov}(x, y) = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{n} \sum_{i=1}^{n} (x_iy_i - \bar{x}\bar{y})\)

Correlation coefficient \(r = \frac{\text{cov}(x, y)}{s_x s_y}\)

Linear regression by least squares

The least squares fit to the linear relationship

\[ y = a + b(x - \bar{x}) \]

is given by

\[ a = \bar{y}, \quad b = \frac{\text{cov}(x, y)}{s_x^2} \]

The corresponding mean square residual is \(s_y^2(1 - r^2)\).