Methods for Differential Equations

Answer all four questions
Blank
1 (i) The equations governing the two competing species \( x \) and \( y \) are
\[
\dot{x} = x(1 - 2x - y), \quad \dot{y} = 2y(1 - y - 2xy),
\]
where \( x(\geq 0) \) and \( y(\geq 0) \) are measured in appropriate units. Find the equilibrium points.

\((3 \text{ marks})\)

(ii) If \( V(x, y) = px^2 + qy^4 \) where \( p \) and \( q \) are constants with \( p > 0 \) and \( q > 0 \), show that \( V(x, y) \) can be a strong Liapunov function for the system
\[
\dot{x} = -2xy^4 - x^5, \quad \dot{y} = x^2y - y^3,
\]
for the equilibrium point \((0,0)\).
State what can be deduced about the nature of this equilibrium point.

\((9 \text{ marks})\)

(iii) Transform \( y'' - \frac{2m}{x} y' + \left[ 1 + \frac{m(m+1)}{x^2} \right] y = 0 \) into normal form, and hence find its general solution.

\((13 \text{ marks})\)
(i) Find the linear approximation to the following system in the neighbourhood of the point \((0,0)\), and determine whether the point \((0,0)\) is stable, asymptotically stable or unstable.

\[
\dot{x} = -x - y - 3x^2 y, \quad \dot{y} = -x - 4y
\]

(5 marks)

(ii) Are the following quadratic functions positive definite, negative definite, or neither?

(a) \(x^2 - xy - y^2\), (b) \(2x^2 - 3xy + 3y^2\).

(6 marks)

(iii) Let

\[
V = \alpha x^2 + \beta x^4 + y^2,
\]

where \(\alpha\) and \(\beta\) are positive constants, and

\[
\dot{x} = y, \quad \dot{y} = -cy - x - x^3,
\]

where \(c\) is a positive constant. Find suitable values of \(\alpha\) and \(\beta\), for which \(V\) is a weak Liapunov function, but not a strong Liapunov function, for the equilibrium point \((0,0)\).

State what can be deduced about the nature of this equilibrium point.

(14 marks)
3 (i) Find all the eigenvalues $\lambda_n \ (n = 1, 2, \ldots)$ with $\lambda_1 < \lambda_2 < \ldots$ of the eigenvalue problem

$$y'' - 2y' + \lambda y = 0$$

$$y(0) = 0, \quad y(1) = 0,$$

and show that the corresponding eigenfunctions are $y_n(x) = B_n e^{x} \sin \alpha_n x$, where $B_n \ (n = 1, 2, \ldots)$ are constants and $\alpha_n$ are the positive roots of $\sin \sqrt{\lambda} - 1 = 0$.

You should consider separately the cases:

(a) $\lambda < 1$; (b) $\lambda > 1$.

(14 marks)

(ii) Show that $x = 0$ is a regular singular point of the differential equation

$$4x^2y'' - 2xy' + (2 + x)y = 0.$$

(5 marks)

Using the Frobenius expansion:

$$y = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \quad a_0 \neq 0,$$

show that the roots of the indicial equation are 1 and $\frac{1}{2}$.

(6 marks)
(i) By means of the substitution \( y = x^\frac{1}{2}z \), show that
\[
y'' + x^2 y = 0 \quad (\ast)
\]
transforms into
\[
x^2 z'' + xz' + (x^4 - \frac{1}{4})z = 0 \quad (\ast\ast)
\]

If \( t = x^2 \), show that \((\ast\ast)\) can be re-written in the form
\[
t^2 \frac{d^2 z}{dt^2} + t \frac{dz}{dt} + \left( t^2 - \frac{1}{16} \right) z = 0,
\]
which is Bessel’s equation of order \( \nu = \frac{1}{4} \).

Hence write down the general solution of \((\ast)\) in terms of Bessel functions.

(ii) Find power series solution about \( x = 0 \) for the following second order ordinary differential equation.
\[
\frac{d^2 y}{dx^2} - x^2 \frac{dy}{dx} + xy = 0
\]