Metric Spaces

Attempt all the questions. The allocation of marks is shown in brackets.

1. (i) Show that a sequence \((x_n)\) in a metric space can have at most one limit. \(\text{(4 marks)}\)

   (ii) Let \(n \geq 1\). Define the taxicab metric \(d_1\) and the maximum or supremum metric \(d_\infty\) on \(\mathbb{R}^n\). \(\text{(2 marks)}\)

      For each of the points \(p = (\frac{7}{8}, \frac{1}{8}, \frac{9}{8})\) and \(q = (\frac{7}{8}, 0, \frac{11}{8})\) of \(\mathbb{R}^3\), determine whether it is in the open ball \(B_1((1, 0, 1), \frac{3}{8})\) for the metric \(d_1\) and whether it is in the open ball \(B_\infty((1, 0, 1), \frac{3}{8})\) for the metric \(d_\infty\). \(\text{(4 marks)}\)

      Which, if any, of your answers would change if the open balls were replaced by the closed balls \(B_1[(1, 0, 1), \frac{3}{8}]\) and \(B_\infty[(1, 0, 1), \frac{3}{8}]\)? \(\text{(2 marks)}\)

   (iii) Let \(a = (a_1, \ldots, a_n), b = (b_1, \ldots, b_n) \in \mathbb{R}^n\). Show that

      \[d_\infty(a, b) \leq d_1(a, b) \leq nd_\infty(a, b)\]

      and deduce that, for \(r > 0\),

      \[B_\infty(a, \frac{r}{n}) \subseteq B_1(a, r) \subseteq B_\infty(a, r)\]. \(\text{(4 marks)}\)

Let \(f : X \to Y\) be a function between metric spaces. Explain, in terms of open balls, what it means for \(f\) to be continuous. \(\text{(3 marks)}\)

Let \(X\) be a metric space and let \(f : \mathbb{R}^n \to \mathbb{R}^n\) be a function. Show that \(f : (\mathbb{R}^n, d_1) \to (\mathbb{R}^n, d_1)\) is continuous if and only if \(f : (\mathbb{R}^n, d_\infty) \to (\mathbb{R}^n, d_\infty)\) is continuous. \(\text{(6 marks)}\)
The metrics $d_1$ and $d_\infty$ on the space $C([0, 1])$ of continuous functions from $[0, 1]$ to $\mathbb{R}$ are given by the rules

\[ d_1(f, g) = \int_0^1 |f(x) - g(x)| \, dx, \quad d_\infty(f, g) = \sup_{0 \leq x \leq 1} |f(x) - g(x)|. \]

(i) Let $(f_n)$ be any sequence of functions in $C([0, 1])$ and let $f \in C([0, 1])$. Show that if $(f_n)$ converges to $f$ in $(C([0, 1]), d_\infty)$ then $(f_n)$ converges to $f$ in $(C([0, 1]), d_1)$. (5 marks)

(ii) For $n \geq 1$, let $f_n \in C([0, 1])$ be given by the rules

\[ f_n(x) = \begin{cases} 
  x^2 + (n - \frac{1}{n})x & \text{if } 0 \leq x \leq \frac{1}{n} \\
  1 & \text{if } \frac{1}{n} \leq x \leq 1.
\end{cases} \]

Let $f \in C([0, 1])$ be given by the rule $f(x) = 1$ for all $x \in [0, 1]$.

(a) Find $d_1(f_n, f)$ and $d_\infty(f_n, f)$ for $n \geq 1$. (7 marks)

Deduce that $(f_n) \to f$ in $(C([0, 1]), d_1)$ but that $(f_n)$ does not converge to $f$ in $(C([0, 1]), d_\infty)$. (2 marks)

(b) Show that the sets

\[ S_1 = \{ g \in C([0, 1]) : g(0) = 0 \} \quad \text{and} \quad S_2 = \{ g \in C([0, 1]) : g(0) \neq 1 \} \]

are not closed in $(C([0, 1]), d_1)$. (3 marks)

Is the set $S_1$ open in $(C([0, 1]), d_1)$? Justify your answer. (4 marks)

(c) Let $\theta : C([0, 1]) \to C([0, 1])$ be the identity function, $\theta(f) = f$ for all $f \in C([0, 1])$. Using (i) and (a), explain why $\theta$ is continuous as a function from $(C([0, 1]), d_\infty)$ to $(C([0, 1]), d_1)$ but not as a function from $(C([0, 1]), d_1)$ to $(C([0, 1]), d_\infty)$. (4 marks)
3 (i) Explain what it means for a subset $A$ of a metric space to be complete. (2 marks)

Show that in a complete metric space $X$ every closed subset of $X$ is complete. (3 marks)

For each of the following subsets $S_i$ of $\mathbb{R}$, determine whether $S_i$ is complete under the usual metric on $\mathbb{R}$. Justify your answers. (7 marks)

(a) $S_1 = (0, 1)$;
(b) $S_2 = \mathbb{R} \setminus \mathbb{Q}$ (the set of irrational numbers);
(c) $S_3 = \mathbb{R} \setminus S_1$.

(ii) Explain what it means for a subset $A$ of a metric space to be compact. (2 marks)

Show that in a compact metric space $X$ every closed subset of $X$ is compact. (3 marks)

For each of the following subsets $A_i$ of $\mathbb{R}^4$, determine whether $A_i$ is compact under the Euclidean metric $d_2$. Justify your answers. (8 marks)

(a) $A_1$ is the closed ball $B[(2, 0, 1, 5), 1]$;
(b) $A_2$ is the complement $\mathbb{R}^4 \setminus B((2, 0, 1, 5), 1)$ of the open ball $B((2, 0, 1, 5), 1)$;
(c) $A_3$ is the intersection of all closed balls of the form $B[(x, x, x, x), 5]$, $0 \leq x \leq 1$.

4 (i) Let $f : X \to X$ be a function from a metric space to itself. Explain what it means for $f$ to be a contraction on $X$. State, without proof, the Contraction Mapping Principle. (4 marks)

Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. State, without proof, a differential criterion for $f$ to be a contraction on $\mathbb{R}$. (2 marks)

Let $a, b$ be positive real numbers. Show that the function $g(x) = ax \cos(bx)$ is not a contraction and that if $ab < 1$ then the function $f(x) = a \sin(bx) + 1$ is a contraction on $\mathbb{R}$. (6 marks)

Show that there exists a unique real number $x$ such that $3 \sin(2x) = 7x - 7$. (3 marks)

Explain briefly how you would calculate successively better approximate values for the number $x$. (Do not attempt the calculation!) (2 marks)

(ii) Let $X$ be a complete metric space and let $g : X \to X$ and $h : X \to X$ be contractions on $X$. Show that the composite function $g \circ h$ is a contraction on $X$. (4 marks)

Explain why each of $g \circ h$ and $h \circ g$ has a unique fixed point in $X$ and let $y, z$ be the unique fixed points of $g \circ h$ and $h \circ g$ respectively. Show that $h(y) = z$ and $g(z) = y$. (4 marks)

End of Question Paper