Answer four questions. If you answer more than four questions, only your best four will be counted.
1 (i) Express the quotient \( \frac{2 + 11i}{3 + 4i} \) in the form \( x + iy \). \((2 \text{ marks})\)

(ii) Express \( \frac{(1 + i)^9}{(1 - \sqrt{3}i)^{11}} \) in the form \( r e^{i\theta} \) with \( r > 0 \) and \( -\pi < \theta \leq \pi \). \((4 \text{ marks})\)

(iii) State, without proof, the triangle inequalities for \( |z + w| \) and \( |z - w| \). \((1 \text{ mark})\)

(a) Show that, if \( |z + 2 + 3i| \leq 1 \), then
\[
3 \leq |2z + 1 + 2i| \leq 7. \quad (3 \text{ marks})
\]

(b) Sketch the set \( S = \{z \in \mathbb{C} : -3 \leq \text{Re}z \leq 3, \text{ and } -3 \leq \text{Im}z \leq 0\} \). Show that for all \( z \in S \),
\[
\left| \frac{\cos z}{e^z} \right| \leq e^3 \cosh 3. \quad (4 \text{ marks})
\]

(iv) Find all the solutions of the following equation:
\[
\cosh z = -3.
\]
You should express your answers in the form \( x + iy \). \((4 \text{ marks})\)

(v) The path \( \gamma \) is the arc of the circle \( |z + 1| = 1 \) from \( 0 \) to \( -2 \) given by \( z = -1 + e^{it} \ (0 \leq t \leq \pi) \). Evaluate
\[
\int_{\gamma} \text{Re}z \, dz, \quad \int_{\gamma} z \sin (z^2) \cos (z^2) \, dz. \quad (7 \text{ marks})
\]
2 (i) Define what is meant by the following two statements:
(a) A function $f$ is **differentiable at the point** $z_0$;
(b) A function $f$ is **analytic in a region** $D$.  \hspace{1cm} (2 marks)

Let
\[ g(z) = \frac{\sinh(z)}{1 + z^5}. \]

Decide where $g$ is analytic giving reasons for your answer.  \hspace{1cm} (4 marks)

(ii) State, without proof, the Cauchy-Riemann equations for a differentiable function.  \hspace{1cm} (1 mark)

Let $h(z) = \text{Re} z + \text{Im} z$ for all $z \in \mathbb{C}$. Prove that the function $h$ is nowhere differentiable.  \hspace{1cm} (3 marks)

(iii) In each of the following cases, determine whether there is a function $k$ analytic on $\mathbb{C}$ with $\text{Re} (k(x + iy)) = u(x, y)$, giving reasons for your answers:

\begin{align*}
(c) \quad u(x, y) &= \cosh x + \cosh y, \\
(d) \quad u(x, y) &= \cosh x \cos y - 2 \sinh x \sin y + 1.
\end{align*}

When $k$ exists, find an explicit expression for $k(z)$ in terms of $z$ and show that you have found all the functions satisfying the conditions.  \hspace{1cm} (9 marks)

(iv) The path $\alpha$ is the semi-circle from the point 1 to the point $-3$, given by $z = -1 + 2e^{it}$ ($0 \leq t \leq \pi$). Show that
\[ \left| \int_{\alpha} \frac{e^z \sin z}{\text{Re}(z + 4)} \, dz \right| \leq 2\pi e \cosh 2. \]  \hspace{1cm} (6 marks)
3 State, without proof, Cauchy’s Theorem and Cauchy’s Integral Formulae for a function and for its derivatives. Your statement should include conditions under which the results are valid. 

(7 marks)

Let \( \gamma \) be the square contour with vertices \( 2, 1 + i, 0, 1 - i \) described in the anti-clockwise direction. Without using the Residue Theorem, evaluate

(i) \( \int_{\gamma} \frac{\cosh z}{z - 1} \, dz \),
(ii) \( \int_{\gamma} \frac{2 \cos z + \cosh z}{z^2 + 2} \, dz \),
(iii) \( \int_{\gamma} \frac{e^z}{4z^2 - 1} \, dz \),
(iv) \( \int_{\gamma} \frac{\sin(\pi z)}{(z - 1)^8} \, dz \).

(14 marks)

Let the contour \( \alpha \) be the circle \( |z - 1| = 2 \) described in the positive direction. Evaluate

\[ \int_{\alpha} [z^3 + \text{Re}(z + 1)] \, dz. \]

(4 marks)
4 (i) Define what is meant by the statement that the power series \( \sum_{n=0}^{\infty} a_n (z - a)^n \) has radius of convergence \( c \).

(1 mark)

Find the radius of convergence of the power series

\[
\sum_{n=0}^{\infty} (-1)^n \frac{z^{3n}}{(3 + 4i)^n}.
\]

(4 marks)

(ii) Show that if the function \( f \) has a zero of order \( k \) at \( \alpha \), then \( \frac{1}{f} \) has a pole of order \( k \) at \( \alpha \).

(5 marks)

For each of the following functions, find all the singularities in \( \mathbb{C} \). Classify these singularities giving reasons for your answers and evaluate the residue at each of them:

(a) \( \frac{1}{e^{i\pi z} \sinh \pi z} \),

(4 marks)

(b) \( (z + 1) \sin \left( \frac{1}{z - 1} \right) \),

(4 marks)

(c) \( \frac{\sin^2(\pi z)}{(z + 1)^2} \),

(3 marks)

(d) \( \frac{\sin(\pi z)}{(z + 1)^6} \).

(4 marks)
5 (i) Let \( \gamma \) be the rectangular contour with vertices \( 5+4i, -5+4i, -5-4i, 5-4i \) described in the anti-clockwise direction. Evaluate
\[
\int_\gamma \frac{\sin z}{1+e^z} \, dz, \quad \int_\gamma z \exp\left(\frac{1}{(z-2)^2}\right) \, dz.
\]
using Cauchy’s Residue Theorem

(ii) Let \( \alpha > 0 \). Prove that
\[
\int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2 + 2x + 5} \, dx = \frac{\pi \cos \alpha}{2e^{2\alpha}}.
\]

Hence, or otherwise, evaluate
\[
\int_{-\infty}^{\infty} \frac{\cos^2 x}{x^2 + 2x + 5} \, dx \quad (4 \text{ marks})
\]

You may assume that
\[
\int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 5} = \frac{\pi}{2}.
\]

End of Question Paper