SCHOOL OF MATHEMATICS AND STATISTICS

Fields

Attempt all the questions. The allocation of marks is shown in brackets.

1. (i) For each of the subsets $J_1, J_2$ of $\mathbb{C}$ specified below determine, with justification, whether it is a subfield of $\mathbb{C}$ (where $i^2 = -1$):
   (a) $J_1 = \{a + bi\sqrt{7} : a, b \in \mathbb{Q}\}, \quad (4 \text{ marks})$
   (b) $J_2 = \{a + bi + ci\sqrt{3} : a, b, c \in \mathbb{Q}\}. \quad (3 \text{ marks})$

(ii) Let $L = \mathbb{Q}(\sqrt{2}, i\sqrt{3})$.
   (a) Show that $L = \mathbb{Q}(2\sqrt{2} + i3\sqrt{3}). \quad (5 \text{ marks})$
   (b) Express the element $\frac{1}{1 + 2\sqrt{2} + i3\sqrt{3}}$ in the form
       $a + b\sqrt{2} + ci\sqrt{3} + di\sqrt{6}$
       for some $a, b, c, d \in \mathbb{Q}. \quad (4 \text{ marks})$

(iii) Let $p$ be a prime number. Prove that
       $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$
       is an irreducible polynomial in $\mathbb{Q}[x]. \quad (9 \text{ marks})$
2  (i) Let $K \subseteq L$ be a field extension. What is meant by saying that an element $b \in L$ is algebraic over $K$? \hspace{1cm} (2 \text{ marks})

(ii) Is the element $b = \sqrt{\sqrt{2} + \sqrt{3}}$ algebraic over $\mathbb{Q}$? Justify your response and if the answer is ‘yes’ then find a polynomial $f(x) \in \mathbb{Q}[x]$ such that $f(b) = 0$. \hspace{1cm} (5 \text{ marks})

(iii) Let $K \subseteq L$ be a field extension. Suppose that an element $c \in L$ is algebraic over $K$.

(a) Give a definition of the minimal polynomial $m(x) \in K[x]$ of the element $c$ over $K$ and prove that it is an irreducible polynomial over $K$. \hspace{1cm} (6 \text{ marks})

(b) Suppose that $n = \deg(m(x))$. Prove that the set of elements $1, c, c^2, \ldots, c^{n-1}$ form a basis for the vector space $K(c)$ over the field $K$. \hspace{1cm} (9 \text{ marks})

(c) Find the minimal polynomial $m(x) \in \mathbb{Q}([\sqrt{2}])[x]$ of the element $c = \sqrt{2} + 2i\sqrt{3}$ over the field $\mathbb{Q}([\sqrt{2}])$. \hspace{1cm} (3 \text{ marks})

3  (i) Define the content of a polynomial and find the content of the polynomial $p = \sum_{n=10}^{100} n!x^n$. \hspace{1cm} (4 \text{ marks})

(ii) State Gauss’s Lemma. \hspace{1cm} (3 \text{ marks})

(iii) Prove Gauss’s Lemma. \hspace{1cm} (9 \text{ marks})

(iv) Show that the polynomial $f(x) = 3x^2 + 2015x + 2$ is an irreducible polynomial over $\mathbb{Q}$. \hspace{1cm} (9 \text{ marks})
Let $B$ be a set of at least two points in the plane $\mathbb{R}^2$ and $P \in \mathbb{R}^2$. What is meant by saying that

(a) $P$ is constructible in one step from $B$, \hspace{1cm} (2 \text{ marks})

(b) $P$ is constructible from $B$. \hspace{1cm} (3 \text{ marks})

(ii) Let $a, b \in \mathbb{R}$ be constructible real numbers. Show that $a - b$ and $a/b$ (provided $b \neq 0$) are also constructible real numbers. \hspace{1cm} (5 \text{ marks})

(iii) (a) Give a criterion for the regular $n$-gon to be constructible. \hspace{1cm} (3 \text{ marks})

(b) Which of the following regular $n$-gons can be constructed (justify your response):

\[ n = 6, 14, 68. \] \hspace{1cm} (5 \text{ marks})

(c) Let $m, n \geq 3$ be natural numbers such that the regular $m$-gon and the regular $n$-gon can be constructed. Show that the regular $2^k n$-gon can be constructed for all $k \geq 1$ and that the regular $mn$-gon can be constructed provided that the numbers $m$ and $n$ are co-prime. \hspace{1cm} (7 \text{ marks})

End of Question Paper