Attempt all the questions. The allocation of marks is shown in brackets.

1. (i) The vector field \( \mathbf{F} \) is given by
\[
\mathbf{F} = (x^2 - 2y)i + (2x - z)j + (y + z^2)k.
\]
Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \), where \( C \) is the unit circle \( x^2 + y^2 = 1, \quad z = 0 \), traversed counterclockwise, starting and finishing at the point \( (1, 0, 0) \). (10 marks)

(ii) Find the Laurent series expansion of
\[
f(z) = \frac{1}{2j - (1 + 2j)z + z^2},
\]
in the region \( 1 < |z| < 2 \). (15 marks)

2. (i) Find all the poles of the function
\[
f(z) = \frac{(z - 1)^3}{z^2(z^2 - 2z + 5)},
\]
and plot them on an Argand diagram. Hence, evaluate the integral \( \oint_C f(z)dz \), writing your solution in the form \( a + jb \), where \( a \) and \( b \) are real, and
(a) \( C \) is the circle \( |z| = 3 \)
(b) \( C \) is the circle \( |z - 4| = 1 \). (15 marks)

(ii) By constructing a suitable contour integral in the complex plane, use the method of residues to evaluate the real integral
\[
I = \int_{-\infty}^{\infty} \frac{1}{5 + 2x + x^2}dx.
\]
(10 marks)
3  (i) Assuming that $z = x + jy$ is a complex number, sketch the regions in the $z$-plane corresponding to (i) $x \leq -2$, (ii) $y \geq x - 3$, (iii) $|z| \geq 4$, and (iv) $|z - 2 + j| \leq 1$.  

(ii) Let $a$ and $b$ be two complex numbers. The mapping $w = az + b$ maps the point $z = -1$ to the point $w = 1 + j$ and the point $z = -1 + j$ to the point $w = 2 + 3j$.

(a) Show that $a = 2 - j$ and $b = 3$.

(b) Using the above values of $a$ and $b$ determine the real-valued functions $u(x, y)$ and $v(x, y)$ given that $w = u + jv$.

(c) Find the image in the $w$-plane of the region $|z| \leq 5$ in the $z$-plane under this mapping.  

(iii) Determine whether the image of the straight line $z = t + 3 + j$, where $t$ is real, under the bilinear mapping

$$w = \frac{z - 2j}{z + j}$$

is a circle or a straight line.  

4  A vector field $\mathbf{A} = \mathbf{A}(x, y, z)$ is given by

$$\mathbf{A} = (4xy - z^3)i + 2x^2j - 3xz^2k.$$  

(i) Calculate div $\mathbf{A}$ and show that curl $\mathbf{A} = 0$.  

(ii) By evaluating both sides, verify that

$$\nabla^2 \mathbf{A} = \text{grad div} \mathbf{A} - \text{curl curl} \mathbf{A}.$$  

(iii) Find a scalar field, $\Phi = \Phi(x, y, z)$, such that

$$\mathbf{A} = \text{grad} \Phi.$$  

End of Question Paper
Formula sheet

- The general formula for the residue at a pole $z_0$, of order $m$ is

\[
\frac{1}{(m-1)! \lim_{z \to z_0} \left\{ \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \right\}}
\]

- Useful identities

\[
\cos^2 \theta = \frac{1 + \cos 2\theta}{2}; \quad \sin m\theta \cos n\theta = \frac{1}{2}[\sin(m+n)\theta + \sin(m-n)\theta]
\]