Answer Question 1 and three other questions. You are advised not to answer more than three of the questions 2 to 5: if you do, only your best three will be counted.

1. (i) Let $\chi$ be a character of the group $(\mathbb{Z}/N\mathbb{Z})^\times$. Define the Dirichlet $L$-series $L(s, \chi)$. Describe a region where $L(s, \chi)$ is analytic and convergent, dependent on $\chi$. (3 marks)

(ii) (a) List all characters of $(\mathbb{Z}/18\mathbb{Z})^\times$. (6 marks)

(b) For the non-trivial real-valued character in your list, show explicitly the corresponding Dirichlet $L$-series does not vanish at $s = 1$. (2 marks)

(c) Verify for all primes $p$:

$$\sum_{\chi} \chi(5)^{-1}\chi(p) = \begin{cases} 
6 & \text{if } p \equiv 5 \mod 18 \\
0 & \text{otherwise}
\end{cases}$$

where the sum is over all characters $\chi$ of $(\mathbb{Z}/18\mathbb{Z})^\times$. (6 marks)

(iii) Using parts (i) and (ii) prove that there are infinitely many primes congruent to 5 mod 18.

(You may assume that for any character $\chi$ the sum $\sum_{p \not\mid 18} \sum_{n \geq 2} \frac{\chi(p)^n}{np^{ns}}$ converges to a finite limit as $s \to 1$ and that $L(1, \chi) \neq 0$.) (8 marks)
(i) Let $p_1, p_2, \ldots, p_n$ be the first $n$ primes. For $x \geq 1$ let $N_n(x)$ be the number of integers $1 \leq k \leq x$ that are divisible only by the primes $p_1, p_2, \ldots, p_n$.

Show that

$$N_n(x) \leq 2^n \sqrt{x}.$$ 

Hence deduce that there are infinitely many primes. \hspace{1cm} (6 marks)

(ii) Consider the following infinite series over all primes:

$$S = \sum_{p} \frac{1}{p}.$$ 

(a) Show that $S$ diverges:

(α) by using the result in part (i). \hspace{1cm} (6 marks)

(β) by using the Euler product expansion of $\zeta(s)$. (You may assume that $\zeta(\sigma) > 0$ for real $\sigma > 1$ and that $\zeta(\sigma) \to \infty$ as $\sigma \to 1^+$.) \hspace{1cm} (7 marks)

(b) Explain why the divergence of $S$ proves that there are infinitely many primes. \hspace{1cm} (2 marks)

(iii) Does $\sum_{p} \frac{1}{p^2}$ converge? Justify your answer. \hspace{1cm} (2 marks)

(iv) Let $n \geq 2$ be a fixed integer. Are there infinitely many primes that are one less than an $n$th power? \hspace{1cm} (2 marks)
(i) Let \( f, g \) be real functions. Define what it means for \( f \) and \( g \) to be asymptotic and state the Prime Number Theorem (PNT). (Your answer should include a definition of the prime counting function \( \pi(x) \).) 

(3 marks)

(ii) Fix a positive integer \( k \). For \( x \geq 1 \) let \( \pi_k(x) \) be the number of primes \( p \) such that \( p^k \leq x \).

(a) Using the PNT show that

\[
\pi_k(x) \sim \frac{k \sqrt{x}}{\ln x}.
\]

(4 marks)

(b) For \( 0 < a < b \) evaluate

\[
\lim_{x \to \infty} \frac{\pi_k(bx)}{\pi_k(ax)}.
\]

Hence, or otherwise, prove that for \( x \) large enough there exists a prime \( p \) such that \( ax < p^k < bx \). 

(8 marks)

(c) Let \( L \) be a positive integer. Using part (b) prove that there are infinitely many primes \( p \) such that the decimal representation of \( p^k \) begins with \( L \). 

(4 marks)

(iii) Let \( m \geq 1 \) be an integer and let \( a \) be an integer coprime to \( m \). For \( x \geq 1 \) let \( \pi_{m,a}(x) \) be the number of primes \( p \leq x \) that are congruent to \( a \mod m \).

Assuming that \( \pi_{m,a} \sim \pi_{m,b} \) for any \( a \) and \( b \) coprime to \( m \) show that

\[
\pi_{m,a}(x) \sim \frac{x}{\phi(m) \ln x}.
\]

(You may use any valid properties of \( \sim \) without proof.) 

(6 marks)
(i) Let $f$ and $g$ be arithmetic functions.
   (a) Define the Dirichlet series $D(s, f)$ and the Dirichlet convolution $f * g$.
   (b) Write down the relationship between the Dirichlet series for $f, g$ and $f * g$. (3 marks)

(ii) (a) If $f$ is completely multiplicative and $g_1, ..., g_n$ are arithmetic functions then show the following for $n \geq 2$:
   $$f(g_1 * g_2 * ... * g_n) = f g_1 * f g_2 * ... * f g_n.$$ (4 marks)

   (b) Hence show that, for $a_1, a_2, ..., a_n \in \mathbb{Z}$:
   $$N_{a_1} * N_{a_2} * ... * N_{a_n} = N_a(N_{a_1-a_n} * N_{a_2-a_n} * ... * N_{a_{n-1}-a_n} * u)$$
   (where $N_a(n) = n^a$ and $u(n) = 1$ for all $n$). (4 marks)

(iii) Write down the Euler product expansion of $D(s, f)$ for multiplicative $f$. Hence show that if $f$ is completely multiplicative then
   $$D(s, f) = \prod_p \left( 1 - \frac{f(p)}{p^s} \right)^{-1}.$$ (4 marks)

(iv) Liouville’s function is defined by
   $$\lambda(n) = \begin{cases} 
   1 & \text{if } n = 1 \\
   (-1)^{a_1+a_2+...+a_k} & \text{if } n \text{ has prime factorisation } p_1^{a_1} p_2^{a_2} ... p_k^{a_k}.
   \end{cases}$$
   (a) Where does $D(s, \lambda)$ converge absolutely? (1 mark)
   (b) Prove that $\lambda$ is completely multiplicative and show that
   $$D(s, \lambda) = \frac{\zeta(2s)}{\zeta(s)}.$$ (7 marks)

   (c) Given an integer $\alpha$ let $\lambda_\alpha$ be the arithmetic function defined by
   $$\lambda_\alpha(n) = \sum_{d|n} d^\alpha \lambda(d).$$ Write $D(s, \lambda_\alpha)$ in terms of the Riemann zeta function. (6 marks)
(i) Define the Riemann zeta function \( \zeta(s) \) as a Dirichlet series and show convergence for \( \text{Re}(s) > 1 \). State the Riemann Hypothesis.  

(7 marks)

(ii) For \( k \geq 0 \) the Bernoulli polynomials \( B_k(x) \) are given by the generating series

\[
\frac{te^{xt}}{e^t-1} = \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!}.
\]

For \( k \geq 1 \) show the following:

(a) \( \frac{d}{dx}(B_k(x)) = kB_{k-1}(x) \),

(b) \( \int_{0}^{1} B_k(x)dx = 0 \).

(6 marks)

(iii) In the interval \([0, 1]\), \( B_3(x) \) has a Fourier expansion of the form

\[
B_3(x) = 12 \sum_{n=1}^{\infty} \frac{\sin 2n\pi x}{(2n\pi)^3}.
\]

Using part (ii), or otherwise, find the Fourier expansion of \( B_4(x) \) in the interval \([0, 1]\). Hence show that \( \zeta(4) = \frac{\pi^4}{90} \). (It is given that \( B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30} \)).

(7 marks)

(iv) For real \( a > 0 \) and \( \text{Re}(s) > 1 \) the Hurwitz zeta function is given by

\[
\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}.
\]

(a) Show that

\[
\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s).
\]

(2 marks)

(b) If \( \chi \) is a mod \( k \) Dirichlet character then show that

\[
L(s, \chi) = k^{-s} \sum_{r=0}^{k} \chi(r) \zeta\left(s, \frac{r}{k}\right).
\]

(3 marks)

End of Question Paper