SCHOOL OF MATHEMATICS AND STATISTICS
Spring Semester
2014–2015

Mathematics Core II
2 hours

Attempt all the questions. The allocation of marks is shown in brackets.

Section A

A1 Solve the following inequalities:
(i) \(|2x - 3| < 7\); (2 marks)
(ii) \(\frac{1}{x} \leq 2x + 1\). (3 marks)

A2 (i) Evaluate the limit \(\lim_{x \to 2} \frac{x^3 - x^2 - x - 2}{x - 2}\) by using l’Hôpital’s Rule. (2 marks)
(ii) Evaluate the same limit by another method. (2 marks)

A3 Let \(f(x, y) = x^2y^2 + 5xy^4\).
(i) Let \(c = f(1, 1)\). What is the value of \(c\)? (1 mark)
(ii) What is the equation of the curve in the surface \(z = f(x, y)\) with constant \(y\)-value and passing through the point \((1, 1, c)\)? (1 mark)
(iii) What is the gradient of this curve at \((1, 1, c)\)? (2 marks)
(iv) Give the equation of the tangent plane to the surface \(z = f(x, y)\) at the point \((1, 1, c)\). (3 marks)
A4. Let $z$ be a function of $u$ and $v$ where $u = 2x - y$ and $v = x - 2y$.

(i) Show that 
\[
\frac{\partial z}{\partial x} + 2\frac{\partial z}{\partial y} = -3\frac{\partial z}{\partial v}.
\]
(2 marks)

(ii) Assuming equality of mixed second-order partial derivatives, show that 
\[
\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial y^2} = 3 \left( \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} \right).
\]
(3 marks)

A5. Let $R$ be the triangular region in the $(x, y)$-plane with vertices $(0, 0)$, $(1, 0)$ and $(1, 2)$. Calculate the integral 
\[
\int \int_R 7x^2y + 2xy^2 \, dx \, dy.
\]
(6 marks)

A6. What is the radius of convergence of 
\[
\sum_{n=1}^{\infty} \frac{(n!)^23^{n+1}}{(2n)!} z^n?
\]
(3 marks)

Section B

B1. For the system of linear equations $Ax = b$, where 
\[
A = \begin{pmatrix} 1 & 1 & -2 \\ 2 & -1 & 5 \\ -3 & -2 & 1 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 1 \\ 2 \\ -7 \end{pmatrix},
\]
form an augmented matrix $(A|b)$ and reduce this matrix to row echelon form. Hence determine the full solution of this system. (4 marks)

B2. Determine $a, b, c$ and $d$ in such a way that 
\[
\begin{pmatrix} \frac{3}{5} & 0 & a \\ b & c & \frac{3}{5} \\ 0 & 1 & d \end{pmatrix}
\]
is an orthogonal matrix. (4 marks)
B3 Show that the determinant

\[
\begin{vmatrix}
 b + c & a & a^3 \\
 c + a & b & b^3 \\
 a + b & c & c^3 \\
\end{vmatrix}
\]

has a factor \((a + b + c)\) and then factorise it completely.  
\((4 \text{ marks})\)

B4 (i) Find the equation of the parabola with a directrix \(x = 0\) and a focus \((6, 0)\).  
\((3 \text{ marks})\)

(ii) Find the equation of the ellipse with vertices \((\pm5, 0)\) and foci \((\pm4, 0)\).  
\((3 \text{ marks})\)

B5 If a point \(P(x_1, y_1)\) lies on the ellipse

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
\]

show that the equation for the tangent at \(P\) is given by

\[
\frac{x_1x}{a^2} + \frac{y_1y}{b^2} = 1.
\]

\((4 \text{ marks})\)

B6 Consider a point \(P\) whose distances to the two fixed points \(A(0, 0)\) and \(B(b, 0)\) satisfy \(AP : PB = 1 : r\), where \(b, r > 0\) and \(r \neq 1\).

(i) By deriving its equation show that the locus of \(P\), that is the set of points with the above property, is a circle.  
\((4 \text{ marks})\)

(ii) Show that

\[
\frac{2}{AB} = \frac{1}{AQ} + \frac{1}{AR'}
\]

where \(Q\) and \(R\) are the intersections of the circle with the \(x\)-axis.  
\((4 \text{ marks})\)

End of Question Paper