Vectors and Mechanics

Attempt all the questions. The allocation of marks is shown in brackets. The total number of marks available is 60.

1 Points $P$ and $Q$ have position vectors \( p = 3i + 7j - 2k \) and \( q = -3i + 2j + 2k \) respectively.

Find:
(i) The position vector of the mid-point of $PQ$;
(ii) The vector $\overrightarrow{PQ}$;
(iii) The parametric vector equation of the line $PQ$.  \( (3 \text{ marks}) \)

2 (i) Given the vectors $a = i - j - k$ and $b = 2i + j + 2k$, find $a \cdot b$.
Hence find the angle between the vectors $a$ and $b$, giving your answer in radians correct to three significant figures.

(ii) Given the vectors $u = i + 2j + 7k$ and $v = i + j - 2k$, find $u \times v$.
Hence find a unit vector perpendicular to both $u$ and $v$.  \( (5 \text{ marks}) \)

3 A line $L$ has parametric vector equation
\[
r = (1 + 2\lambda) \, i + (2 - \lambda) \, j + (\lambda - 5) \, k.
\]
A plane $\Pi$ has vector equation
\[
r \cdot (-2i + j - k) = 2.
\]

(i) Find the point of intersection (if any) of the line $L$ and the plane $\Pi$.

(ii) Explain why the line $L$ is perpendicular to the plane $\Pi$.  \( (5 \text{ marks}) \)
4 A particle is projected from the origin $O$ with speed $V$ at an angle $\theta$ above the horizontal. Air resistance can be ignored.

If the horizontal and vertical displacements of the particle at time $t$ are $x$ and $z$ respectively, write down expressions for $x$ and $z$ in terms of $t$, $V$, $\theta$ and the acceleration due to gravity $g$.

A stone is thrown from a height of 1.5 m above flat ground, at an angle $\pi/4$ above the horizontal. It lands on the ground at a horizontal distance of 30 m from the point of projection. Ignoring air resistance and taking the acceleration due to gravity to be $g = 9.8 \text{ m s}^{-2}$, find the speed of the stone when launched, giving your answer correct to two significant figures. \hspace{1cm} (5 marks)

5 At time $t$, the position vector of a moving particle of mass $M$ is $r(t)$, where

$$r(t) = 3at \hat{i} + bt^4 \hat{j}$$

and $a$ and $b$ are positive constants.

Find:

(i) The velocity and acceleration of the particle at time $t$;

(ii) The total force $\mathbf{F}$ acting on the particle at time $t$;

(iii) The kinetic energy of the particle at time $t$;

(iv) The work done by the force $\mathbf{F}$ on the particle during the period from $t = 0$ to $t = T$;

(v) The impulse of the force $\mathbf{F}$ on the particle during the period from $t = 0$ to $t = T$. \hspace{1cm} (7 marks)

6 At time $t = 0$ a particle $A$ is at the origin and a particle $B$ is at the point with position vector $5\hat{i} - 10\hat{j} - 12\hat{k}$ m.

Particle $A$ moves with constant velocity $2\hat{i}$ m s$^{-1}$ and particle $B$ moves with constant velocity $4\hat{i} + 4\hat{j} + 5\hat{k}$ m s$^{-1}$.

Show that the least distance between $A$ and $B$ in the subsequent motion is $\sqrt{89}$ m. \hspace{1cm} (4 marks)
A car of mass $M$ is travelling without slipping round a rough bend of radius $R$ which is banked at an angle $\alpha$ to the horizontal.

Draw clear diagrams showing the forces on the car perpendicular to the driving direction if the car is travelling (a) at the minimum possible speed; (b) at the maximum possible speed. (3 marks)

A particle of mass $m$ moves along the $x$-axis with initial speed $u$.

The only force acting on the particle is a resistance force of magnitude $mkv^2$, where $v$ is the speed of the particle and $k$ is a constant.

Show that the equation of motion of the particle takes the form

$$\frac{1}{v} \frac{dv}{dx} = -k.$$

Find the speed of the particle when it has travelled a distance $x = \frac{\ln 2}{2k}$.

At this instant the particle collides and coalesces with a stationary particle of equal mass $m$. Find the speed of the new particle immediately after the collision. (7 marks)

Two particles $C$ and $D$, of masses $m$ and $m/2$ respectively, are attached to the lower end of a light elastic string of stiffness $9mg/L$ where $g$ is the acceleration due to gravity and $L$ is the natural length of the string. The upper end $O$ of the string is fixed.

Find the extension $y$ of the string when the particles hang in equilibrium.

At time $t = 0$ the particle $D$ falls off the end of the string. In the subsequent motion the distance $OC$ is $x + \frac{10L}{9}$. Show that

$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where $\omega$ is a positive constant which you should find in terms of $g$ and $L$.

Find $x(t)$ in the subsequent motion. (8 marks)
A particle $P$ of mass $M_1$ is attached to one end of a light inextendible string. The other end of the string is attached to a second particle $Q$, of mass $M_2$. The particle $Q$ is resting on a rough plane inclined at an angle $\alpha$ to the horizontal. The coefficient of friction between the particle $Q$ and the inclined plane is $\tan \lambda$. Air resistance can be ignored.

The string passes over a smooth pulley $S$ at the end of the plane. The pulley $S$ exerts no forces on the system. The angle between the string $SP$ and the downwards vertical is $\theta$. The whole system is shown in the diagram below:

![Diagram of the system with particles $P$ and $Q$ and pulley $S$.]

The particle $Q$ is at rest. The particle $P$ moves on an arc of a vertical circle with $S$ at the centre so that the distance $SP = L$ where $L$ is a constant. During the motion of $P$ the angle $\theta$ varies between $-\beta$ and $\beta$.

(i) Draw a clear diagram showing the forces on the particles $P$ and $Q$.

(ii) Show that the particle $Q$ remains at rest if $T$, the magnitude of the tension in the string, satisfies

$$T \leq M_2 g \frac{\sin (\alpha + \lambda)}{\cos \lambda}.$$ 

(iii) By considering the motion of the particle $P$, show that the tension in the string has magnitude

$$T = M_1 g [3 \cos \theta - 2 \cos \beta].$$

Hence find the maximum value of $T$ during the motion of $P$.

(iv) Deduce that $Q$ can remain at rest while $P$ is moving if

$$M_1 \leq M_2 \frac{\sin (\alpha + \lambda)}{\cos \lambda [3 - 2 \cos \beta]}.$$ 

(13 marks)

End of Question Paper