SCHOOL OF MATHEMATICS AND STATISTICS

MAS151 Civil Engineering Mathematics

3 hours

Attempt ALL questions.
Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.

Section A

A1 Let \( f(x) = \frac{x + 2}{x - 3} \). Sketch the curve \( y = f(x) \).

A2 Let \( f(x) = \frac{1}{2} e^{x^3+1} \). Find \( f^{-1}(x) \) and state its domain and range.

A3 If \( f(x, y) = x \ln(x^2 + y^2) \), find \( \frac{\partial f}{\partial x} \), \( \frac{\partial f}{\partial y} \) and \( \frac{\partial^2 f}{\partial x \partial y} \).

A4 Use l’Hôpital’s Rule to evaluate \( \lim_{x \to 0} \left( \frac{x \sin x}{\sinh^2 x} \right) \).

A5 Find all the complex numbers \( z \) for which \( |z - 1| = \sqrt{3} \) and \( z \bar{z} = 4 \), where \( \bar{z} \) is the complex conjugate of \( z \).
A6 Find the value of \( t \) for which \( \mathbf{a} = (3, -1, 2) \) and \( \mathbf{b} = (4, 2, t) \) are perpendicular. For this value of \( t \), evaluate \( \mathbf{a} \times \mathbf{b} \) and find a unit vector in the direction of \( \mathbf{a} \times \mathbf{b} \).

A7 Find the definite integral \( \int_{1}^{e} x(\ln x)^2 dx \) using integration by parts.

A8 Find the indefinite integral \( \int \frac{1}{\sqrt{4x - x^2}} dx \).

A9 Let \( A = \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix} \) and \( B = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \). For each of \( A^{-1}B \) and \( B^{-1}A \), either calculate it or say why it doesn’t exist.

A10 Find the particular solution of the differential equation

\[
(1 + x^2) \frac{dy}{dx} = 2xy + 2x
\]

for which \( y = 1 \) when \( x = 0 \).

A11 For which real values of \( \alpha \) does the system of linear equations below have infinitely many solutions for \( x, y \) and \( z \)? You do not need to find \( x, y \) and \( z \).

\[
\begin{align*}
2x + \alpha y &= 0 \\
x - y + \alpha z &= 0 \\
x + 3y + z &= 0.
\end{align*}
\]

A12 Find the values of \( k \) for which \( y = e^{kx} \) is a solution to

\[
\frac{d^3 y}{dx^3} - 4 \frac{d^2 y}{dx^2} + \frac{dy}{dx} + 6y = 0.
\]
Section B

B1 Find the Maclaurin expansion of \( f(x) = \sin^{-1} x \) up to and including the term involving \( x^3 \). By considering the expansion of \((1 + y)^{\frac{3}{2}}\) or otherwise, find the Maclaurin expansion of \((1 - \sin^{-1} x)^{\frac{3}{2}}\) up to and including the \( x^3 \)-term.

B2 Find and classify the stationary points of \( f(x, y) = x^3 + 4xy - 2y^2 - 7x \).

B3 (i) Let \( z = \cos \theta + i \sin \theta \). By calculating \( z^5 \) in two different ways, show that \( \cos(5\theta) = 16 \cos^5 \theta - 20 \cos^3 \theta + 5 \cos \theta \).

(ii) Use part (i) to show that \( \cos \left( \frac{3\pi}{10} \right) \) is a solution to \( 16x^4 - 20x^2 + 5 = 0 \), and hence find its precise value (expressed using square-roots, not as a decimal).

(You may use without justification that \( \cos \left( \frac{3\pi}{10} \right) \) is the smallest positive root of this equation.)

B4 Particles A and B have position vectors \( \mathbf{r}_A \) and \( \mathbf{r}_B \) at time \( t \) given by

\[
\mathbf{r}_A = (\cos(\pi t), \sin(\pi t), t)
\]

\[
\mathbf{r}_B = (1 - 2t, 0, t^2).
\]

(i) Show that at \( t = 0 \), particles A and B are in the same position and that the velocity of A is perpendicular to the velocity of B.

(ii) Determine whether the particles collide at some value of \( t > 0 \).

(iii) Describe the path taken by particle A assuming its path is unaffected by any collision with particle B, and draw a rough sketch.

B5 Let \( t = \tanh(x) \). Show that \( \cosh(2x) = (1 + t^2)/(1 - t^2) \), \( \sinh(2x) = 2t/(1 - t^2) \) and \( \frac{dx}{dt} = \frac{1}{1 - t^2} \). Hence show that

\[
\int \frac{dx}{\sinh(2x)} = \frac{1}{2} \ln |\tanh x| + c
\]

and find \( \int \frac{dx}{\cosh(2x)} \).

( Note: there are formulas for hyperbolic functions on the formula sheet.)
B6 Recall that a 2 × 2 matrix $A$ represents a transformation of the plane: given a point with coordinates $(x, y)$, the new coordinates are given by $A \begin{pmatrix} x \\ y \end{pmatrix}$.

By working out their effects on the square with corners at $(1, 1)$, $(1, -1)$, $(-1, 1)$ and $(-1, -1)$ (or otherwise), describe the transformations represented by the matrices below.

\[
A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad A_4 = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}
\]

B7 Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$. Find all eigenvalues and eigenvectors of $A$.

B8 Find the general solution to the differential equation

\[ y'' - 4y' + 4y = e^x \cos x. \]