Section A

A1  A plane is given by the equation

\[ 3x + 2y + 4z = 21 \]

and a line by the equation \( \mathbf{r} = (1, 2, 3) + \lambda (1, 2, \mu) \), where \( \lambda \) is a real parameter and \( \mu \) is a constant.

(i) Find \( \mu \) so that the line does not intersect the plane. \( (4 \text{ marks}) \)

(ii) Using the value of \( \mu \) you found in part (i), calculate the distance of the line to the plane. \( (4 \text{ marks}) \)

(iii) Find the direction of the line of intersection of the two planes \( 3x + 9y - 3z = 15 \) and \( 3(x - y) + 2z = \pi \). \( (3 \text{ marks}) \)

A2  By explicit calculation, show that for a well-behaved vector field \( \mathbf{F} \) the following identity holds: \( \nabla \cdot (\nabla \times \mathbf{F}) = 0. \) \( (5 \text{ marks}) \)
Stokes’ theorem may be written:

\[ \oint_{C} \mathbf{G} \cdot d\mathbf{r} = \int_{S} (\nabla \times \mathbf{G}) \cdot \mathbf{n} \, dS \]

Indicate whether the following statements about Stokes’ theorem, as expressed here, are true or false

(i) The term \((\nabla \times \mathbf{G})\) is the curl of the vector field \(\mathbf{G}\).

(ii) The surface \(S\) is surrounded by a closed line \(C\).

(iii) \(\mathbf{n}\) is a unit vector parallel with the boundary \(C\).

(iv) \(\int_{S} dS\) is a surface integral, over the surface \(S\).

(4 marks)

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Section B

B1 (i) Consider the function

\[ f(x, y) = \tan^{-1} \frac{y}{x} \]

Show that this equation obeys

\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \]

Hint: \(\tan^{-1}\) (also called arctan) is the inverse function of the \(\tan\)-function and you are given that

\[ \frac{d\tan^{-1} u}{du} = \frac{1}{1 + u^2}. \]

(8 marks)

(ii) A scalar function is given as

\[ \phi(x, y, z) = x^3 - zy \cos(x - z). \]

(a) Calculate the gradient of \(\phi(x, y, z)\), i.e. calculate \(\nabla \phi\).

(b) Using your result, calculate the divergence of \(\mathbf{V}\).

(c) Find the directional derivative of \(\phi\) at the point \((0, 0, -\pi)\) in the \(y\) direction.

(3 marks)  
(3 marks)  
(6 marks)
B2 (i) A vector field is given by

\[ \mathbf{V} = r^n \hat{r} + \left( a + \frac{b}{r} \right) \hat{\theta} + cz \hat{z} \]

in cylindrical polar coordinates, where \( a, b \) and \( c \) are positive constants. Calculate the divergence and curl of the vector field, given that the divergence and curl may be expressed in cylindrical coordinates as

\[ \nabla \cdot \mathbf{V} = \frac{1}{r} \frac{\partial}{\partial r} (r V_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (V_\theta) + \frac{\partial}{\partial z} (V_z) \]

and

\[ \nabla \times \mathbf{V} = \frac{1}{r} \begin{vmatrix} \hat{r} & r \hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ V_r & r V_\theta & V_z \end{vmatrix} \]

respectively. (10 marks)

(ii) A hollow cylinder occupies the region \( a \leq r \leq b, 0 \leq \theta \leq 2\pi, 0 \leq z < a \), where \( (r, \theta, z) \) are cylindrical polar coordinates and \( a \) and \( b \) are positive constants. The mass density of the cylinder is given by

\[ \rho = \frac{\rho_0}{ar} z^2, \quad (1) \]

where \( \rho_0 \) is a positive constant. Write down the volume element \( dV \) in cylindrical polar coordinates and find the mass of the cylinder. (10 marks)

B3 (i) A sphere of radius \( a \) is charged with charge density \( q_0 r^3/a^3 \), where \( r \) is the distance from the centre of the sphere, located at the origin of co-ordinates, and \( q_0 \) is a constant. Find the total charge of the sphere. (10 marks)

(ii) A magnetic field is given, in cylindrical polar coordinates \( (r, \theta, z) \), as \( \mathbf{H} = H_0 r^2 \hat{\theta} / a^2 \), with \( r \leq a \), where \( H_0 \) and \( a \) are positive constants. The magnetic field vanishes for \( r > a \). Evaluate

\[ \oint_C \mathbf{H} \cdot d\mathbf{r}, \]

where \( C \) is the circle \( z = 0, \ r = R \), described in the anticlockwise sense for \( R < a \). (10 marks)

End of Question Paper