SCHOOL OF MATHEMATICS AND STATISTICS  

MAS222 Differential Equations  

2.5 hours

Attempt all the questions. The allocation of marks is shown in brackets.

1. (i) Write the general planar first order autonomous system for the following equation

\[ \ddot{x} + \omega^2 x = 0, \]

where \( \omega \) is a constant.  

(2 marks)

Find the equilibrium point and its nature.  

(5 marks)

Sketch the phase portrait for all three cases: (a) \( \omega < 1 \), (b) \( \omega > 1 \), and (c) \( \omega = 1 \).  

(5 marks)

(ii) Find the equilibrium points of the following system

\[ \dot{x} = x(3 - x - 2y), \quad \dot{y} = y(x - 1). \]

Using the eigenvalues, classify each of the equilibrium points.  

(9 marks)

Sketch the phase portrait.  

(4 marks)
(i) Find the general solution (up to first 5 terms) of
\[ y'' + xy' + y = 0 \]
around \( x_0 = 1 \), using the power series method. \( (12 \text{ marks}) \)

(ii) Write down the self-adjoint form of the differential equation
\[ y'' - 4y' + \mu y = 0, \quad 0 \leq x \leq 1. \]
The above differential equation is an eigenvalue problem for the constant \( \mu \) with \( y(0) = 0 \) and \( y'(1) = 0 \).

State the orthogonality relation (in the form of a definite integral) satisfied by \( y_m \) and \( y_n \), the eigenfunctions associated with the eigenvalues \( \mu_m \) and \( \mu_n \) respectively. \( (5 \text{ marks}) \)

(iii) It is given that \( x = 0 \) is a singular point of the differential equation:
\[ x^2 y'' - (2 + x^2) y = 0. \]
Show that \( x = 0 \) is a regular singular point of the differential equation. \( (2 \text{ marks}) \)

Use the Frobenius series expansion to show that the roots of the indicial equation are 2 and -1. \( (6 \text{ marks}) \)
(i) The function $u(x,t)$ satisfies the heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2},$$

in the domain $0 < x < \pi$, $t > 0$, where $k$ is a positive constant.

(a) Show that separable solutions of the form $X(x)T(t)$ satisfy the relation

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \alpha,$$

where $\alpha$ is a constant. Explain why $\alpha$ is a constant. \hspace{1cm} (2 marks)

(b) Assuming that $\alpha = -s^2$ ($s$ is a positive real number), we have

$$X(x) = A \cos sx + B \sin sx,$$

where $A$ and $B$ are arbitrary constants. Determine the general solution for $u(x,t)$ with boundary conditions

$$u(0,t) = u(\pi,t) = 0 \text{ for } t \geq 0.$$

You may assume there are only trivial separable solutions when $\alpha \geq 0$. \hspace{1cm} (8 marks)

(c) Then find the solution for initial condition $u(x,0) = 4 \sin(3x)$.

\hspace{1cm} (3 marks)

(ii) Find the solution for the following inhomogeneous heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + e^{-kt} \sin(2x), \quad 0 < x < \pi, \quad t > 0$$

with the same boundary and initial conditions as in (i):

$$u(0,t) = u(\pi,t) = 0 \text{ for } t \geq 0, \quad \text{and} \quad u(x,0) = 4 \sin(3x) \text{ for } 0 < x < \pi.$$

$k$ is a positive constant. \hspace{1cm} (12 marks)
4 (i) Find a set of characteristic coordinates $\xi(x, t)$ and $\eta(x, t)$ for the following equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} - 6 \frac{\partial^2 u}{\partial t^2} + x \frac{\partial u}{\partial t} = x^2.$$  

Note: find the characteristic coordinates only. You do NOT need to simplify the PDE. \hspace{2cm} (6 marks)

(ii) Find the solution for the following first order partial differential equation

$$\frac{\partial u}{\partial t} + 2x \frac{\partial u}{\partial x} = -u, \quad u = u(x, t),$$

with $u(x, 0) = \sin x$. \hspace{2cm} (9 marks)

(iii) Consider the heat equation for $u(x, t)$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

in the domain $-1 \leq x \leq 1$, with boundary conditions

$$u(-1, t) = 0, \quad -3u(1, t) + \frac{\partial u(x, t)}{\partial x} \bigg|_{x=1} = 0.$$  

For a separable solution $u(x, t) = X(x)T(t)$, you are given that

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{kT(t)} = \alpha,$$

where $\alpha$ is a constant. Assuming $\alpha = -s^2$ ($s > 0$ is a real number), show that $s$ must satisfy the following relation when the solution is non-trivial:

$$s \cos(2s) = 3 \sin(2s).$$  

(10 marks)

End of Question Paper