Marks will be awarded for your best **FOUR** answers. The marks awarded to each question or section of question are shown in italics.
1 (a) A function $f(x)$ is defined for $-\infty < x < \infty$ by

$$f(x) = e^{-|x|}.$$ 

Show that the Fourier transform, $\hat{f}(k)$, of $f(x)$ for real $k$ is given by

$$\hat{f}(k) = \frac{2}{k^2 + 1}. \quad (6 \text{ marks})$$

(b) A function $g(x)$ is defined for $-\infty < x < \infty$ by

$$g(x) = \sin x.$$ 

Find the Fourier transform, $\hat{g}(k)$, of $g(x)$ for real $k$. \quad (4 \text{ marks})

You may assume that

$$\int_{-\infty}^{\infty} e^{ikx} \, dx = 2\pi \delta(k).$$

(c) By using the inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k) \, dk,$$

or otherwise, show that

$$\mathcal{F}\{f(x)g(x)\} = \frac{1}{2\pi} \left( \hat{f} \ast \hat{g} \right)(k),$$

where $\mathcal{F}$ denotes the Fourier transform, and the convolution $\left( \hat{f} \ast \hat{g} \right)(k)$ is defined by

$$\left( \hat{f} \ast \hat{g} \right)(k) = \int_{-\infty}^{\infty} \hat{f}(s) \hat{g}(k-s) \, ds. \quad (9 \text{ marks})$$

(d) Using the results of parts (a), (b) and (c) show that

$$\mathcal{F}\{e^{-|x|} \sin x\} = i \left[ \frac{1}{(k-1)^2 + 1} - \frac{1}{(k+1)^2 + 1} \right]$$

for real $k. \quad (6 \text{ marks})$
The Laplace transform of a function $f(t)$ is defined by

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt.$$ 

(a) By using the change of variables $t^{1/2} = u$, show that for $\text{Re} \, s > 0$

$$\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi} s^{-1/2}. \quad (3 \text{ marks})$$

You may assume that \( \int_0^\infty e^{-su^2} \, du = \frac{1}{2} \sqrt{\pi/s} \) for $\text{Re} \, s > 0$. 

Hence, by integrating by parts, show that, if $n$ is a positive integer and $\text{Re} \, s > 0$,

$$\mathcal{L}\{t^{n-1/2}\} = \sqrt{\pi} \frac{1}{2} \frac{3}{2} \cdots \left( n - \frac{3}{2} \right) \left( n - \frac{1}{2} \right) s^{-(n+\frac{1}{2})}. \quad (7 \text{ marks})$$

(b) Find the Laplace transform of $\cos \omega t$ for $\text{Re} \, s > 0$. \quad (4 \text{ marks})

(c) If $F(t)$ is defined for $t > 0$ by

$$F(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau,$$

show that

$$\mathcal{L}\{F(t)\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\}. \quad (5 \text{ marks})$$

(d) Use the results of parts (a), (b) and (c) to show that the inverse Laplace transform of

$$\frac{s^{-5/2}}{s^2 + 1}$$

is

$$\frac{8}{15\sqrt{\pi}} \int_0^t \tau^{5/2} \cos(t - \tau) \, d\tau. \quad (6 \text{ marks})$$
The function $y(x)$ satisfies the ordinary differential equation
\[ x^2y'' - 2xy' + 2y = x^3e^{-x} \tag{1} \]
in $-1 < x < 2$, with $y(-1) = y(2) = 0$, where $y' = \frac{dy}{dx}$ etc.

(a) By trying $y = x^n$, find the independent solutions of
\[ x^2y'' - 2xy' + 2y = 0. \tag{4 \text{ marks}} \]

(b) Given that Green’s function $G(x; \xi)$ for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\frac{\partial G}{\partial x}$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that
\[
G(x; \xi) = \begin{cases} 
\frac{\xi - 2}{3\xi^3}(x^2 + x) & -1 < x < \xi, \\
\frac{\xi + 1}{3\xi^3}(x^2 - 2x) & \xi < x \leq 2. 
\end{cases} \tag{15 \text{ marks}}
\]

(c) Using Green’s function, show that the solution to equation (1) which satisfies the boundary conditions given at the beginning of the question is
\[
y(x) = xe^{-x}\frac{1}{3}(2e+e^{-2})x + \frac{1}{3}(e-e^{-2})x^2. \tag{6 \text{ marks}}
\]

\[
\int_{-1}^{x} (\xi + 1)e^{-\xi} \, d\xi = e - (2 + x)e^{-x} \quad \text{and} \quad \int_{x}^{2} (\xi - 2)e^{-\xi} \, d\xi = (x - 1)e^{-x} - e^{-2}.
\]

Consider the equation
\[
(1 + \epsilon)x^2 - 2x - 3 = 0, \tag{*}
\]
where $\epsilon$ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation (\star) can be written as
\[ x = x_0 + \epsilon x_1 + \epsilon^2 x_2 + \cdots, \]
where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to derive the two solutions to equation (\star), correct to order $\epsilon^2$ as $\epsilon \to 0$. \tag{18 \text{ marks}}

(b) Find the exact solutions of (\star), and show that their expansions agree with your results from part (a). \tag{7 \text{ marks}}
The complementary error function, \( \text{erfc}(x) \), is defined for \( x > 0 \) by

\[
\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2} \, dt.
\]

By changing variables, show that

\[
\text{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \int_0^\infty e^{-xv} e^{-v^2/4} \, dv. \quad (3 \text{ marks})
\]

Expand \( e^{-v^2/4} \) in powers of \( v \) and change variables by \( u^2 = xv \). Hence, defining

\[
I_n = \int_0^\infty u^{2n+1} e^{-u^2} \, du,
\]

show that

\[
\sqrt{\pi} x e^{x^2} \text{erfc}(x) \sim 2 \sum_{k=0}^\infty \frac{(-1)^k}{k!(4x^2)^k} I_{2k} \quad \text{as } x \to \infty. \quad (7 \text{ marks})
\]

Show that \( I_n = nI_{n-1} \) for \( n > 0 \), and hence find an asymptotic expansion for \( \text{erfc}(x) \) as \( x \to \infty \). \quad (10 \text{ marks})

For the second term in the asymptotic expansion to make less than a 1\% relative change to the first term, show that \( x > \sqrt{50} \). \quad (5 \text{ marks})

End of Question Paper