1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic.

(a) \( \frac{\partial u}{\partial y} + 3 \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial y^2} = u \)  
\( (1 \text{ mark}) \)

(b) \( \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + 3 = -\frac{\partial u^2}{\partial y^2} - 2u \)  
\( (1 \text{ mark}) \)

(c) \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = 5 \)  
\( (1 \text{ mark}) \)

(ii) Discuss the benefits/drawbacks of the three systems of numerical solutions, i.e. implicit, explicit and Crank-Nicolson.  
\( (6 \text{ marks}) \)

(iii) Consider the following differential equation

\[ 3 \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial^2 u}{\partial x^2} + 42u = 0, \]

By using the explicit difference scheme, solve the differential equation between \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) for \( \Delta x = h = \frac{1}{3}, \quad \Delta y = k = \frac{1}{2} \), and the following boundary conditions

\[ u(1, y) = 0, \quad u(x, 1) = 1 - x^2, \]

under the assumption that the solution is symmetric across both the \( x \) and \( y \) axes, i.e. \( u(x, y) = u(-x, y) = u(x, -y) \).  
\( (15 \text{ marks}) \)

(iv) State the order of the error.  
\( (1 \text{ mark}) \)
2 (i) Discuss the advantages/disadvantages of LU factorisation over ‘traditional’ matrix inversion.

(2 marks)

(ii) Determine the L and U matrices for the following system

\[ Ax = b, \quad A = \begin{bmatrix} 8 & 2 & 9 \\ 4 & 9 & 4 \\ 6 & 7 & 9 \end{bmatrix}, \quad b = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix} \]

(5 marks)

(iii) Determine \( L^{-1} \) and \( U^{-1} \)

(4 marks)

(iv) Hence find the column vector \( x \)

(3 marks)

(v) Find the LU decomposition for the following tri-diagonal matrix

\[ M = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ 0 & 1 & 3 & 2 & 0 \\ 0 & 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix} \]

(6 marks)

(vi) Using your results from part (v), or otherwise, show \( M^{-1} \) to be

\[ M^{-1} = \frac{1}{63} \begin{bmatrix} 31 & -30 & 28 & -24 & 16 \\ -15 & 45 & -42 & 36 & -24 \\ 7 & -21 & 49 & -42 & 28 \\ -3 & 9 & -21 & 45 & -30 \\ 1 & -3 & 7 & -15 & 31 \end{bmatrix} \]

(5 marks)

3 Consider the following expression

\[ f(x, y) = x^2 + \frac{4}{3}xy + y^2 + 6 \quad \text{(1)} \]

(i) State the four expressions which allow for analytical classification of stationary points.

(4 marks)

(ii) Find and classify all stationary points of Equation (1).

(3 marks)

(iii) Apply the method of steepest descent, for one iteration, to Equation (1) starting from the point (1,1).

(8 marks)

(iv) Determine the Newton step for the function given in Equation (1) from a starting point of your choice.

(7 marks)

(v) Discuss the implications of your answer in part (iv).

(3 marks)
4 
(i) Write down the three properties of cubic splines and why they are important. (3 marks)

(ii) Write down the values for $\sigma_0$ and $\sigma_n$ under the assumption of

$$f''(x_0) = 0, \quad f'(x_n) = 0$$

(2 marks)

(iii) Using the conditions derived in (ii), determine the cubic spline between the following data points

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>$\pi/3$</th>
<th>$2\pi/3$</th>
<th>$\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>$1/2$</td>
<td>$-1/2$</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

(18 marks)

(iv) Determine the value of the cubic spline at $f(5\pi/6)$. (2 marks)

5 
(i) In a dynamic network problem, describe how you would represent the relationship between isolated nodes. (2 marks)

(ii) In a dynamic network problem, describe how you would represent the relationship between stage jumping nodes. (2 marks)

(iii) A ship is to be loaded with a selection of goods of 3 types. The total weight of the load must not exceed 20 tonnes. The goods have weights and values as shown in the following table:

<table>
<thead>
<tr>
<th>Type</th>
<th>Weight</th>
<th>Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>10</td>
</tr>
<tr>
<td>2</td>
<td>5</td>
<td>24</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>37</td>
</tr>
</tbody>
</table>

Using the dynamic programming algorithm, construct an appropriate table to find the combination of goods which gives the highest value for the load on the ship. (21 marks)

End of Question Paper
Formulae Sheet

Notation:

\[ U(x_i, t_j) \equiv U_{ij} \]

**Forward difference formula for \( \partial U / \partial t \):**

\[
\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}
\]

**Backward difference formula for \( \partial U / \partial t \):**

\[
\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}
\]

**Central difference formula for \( \partial U / \partial x \):**

\[
\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}
\]

**Central difference formula for \( \partial^2 U / \partial x^2 \):**

\[
\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}
\]