1. (i) Let $G$ be a group. Define $\text{Aut}(G)$ and prove that it is a subgroup of the group $S(G)$ of all bijections $f : G \rightarrow G$. [You may assume without proof that $S(G)$ is a group under composition of functions.]

(ii) Let $G$ be a group. Prove that for $a \in G$ the map $\omega_a : G \rightarrow G$ defined by

$$x \mapsto \omega_a(x) := axa^{-1}$$

is an element of $\text{Aut}(G)$ and that the map $\omega_a : G \rightarrow \text{Aut}(G)$ given by $a \mapsto \omega_a$ is a homomorphism of groups.

(iii) (a) Prove that $\text{Aut}(\mathbb{Z}/n\mathbb{Z}) \cong (\mathbb{Z}/n\mathbb{Z})^*$. [You may assume that $l_a : \mathbb{Z}/n\mathbb{Z} \rightarrow \text{Hom}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$ defined by $a \mapsto l_a$ is a bijection, where $l_a(x) = ax$ for all $x \in \mathbb{Z}/n\mathbb{Z}$.

(b) Express $\text{Aut}(\mathbb{Z}/12\mathbb{Z})$ as a direct product of cyclic groups of prime power order.

(iv) Let $Q = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group where $1$ is the identity of $Q$ and the multiplication is given by the rules:

$$i^2 = j^2 = k^2 = -1, \ (-1)a = a(-1) = -a \text{ for all } a \in Q,$$

$$(-1)^2 = 1, \ ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j.$$

By considering the order of its elements describe 24 different automorphisms of $Q$. 


(i) Define the centre $Z(G)$ of a group $G$ and prove that it is a normal subgroup. 

(4 marks)

(ii) Give one example each of groups $G_1, G_2, G_3$ with

$Z(G_1) = \{e\}$, $Z(G_2) = G_2$ and $\{e\} \subseteq Z(G_3) \subseteq G_3$.

(3 marks)

(iii) (a) Define the special orthogonal group $SO_2$ and the elements $R_\theta, S_\theta$ of $O_2$.

(3 marks)

(b) By multiplying out matrices (and quoting relevant trigonometric identities) show that $S_\theta S_\phi = R_{\theta - \phi}$.

(2 marks)

(c) For $n > 2$ let $D_n = \{R^{i}_{2\pi/n}S^j_0 \text{ for } i = 1 \ldots n \text{ and } j = 0, 1\}$. Determine $Z(D_n)$, distinguishing between $n$ even and odd. You may use the identities $R_\theta R_\phi = R_{\theta + \phi}$, $R_\theta S_\phi = S_{\theta + \phi}$ and $S_\theta R_\phi = S_{\theta - \phi}$ without proving them.

(5 marks)

(iv) Let $F_3 = \mathbb{Z}/3\mathbb{Z}$ be the field with 3 elements.

(a) Calculate the order of the group $SL_3(F_3)$.

(3 marks)

(b) Prove that the centre of $SL_3(F_3)$ is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \right\}.$$ 

(5 marks)
3 (i) (a) Give the definition of the action of a group $G$ on a set $X$. 

(3 marks)

(b) Given a homomorphism $\phi : G \to S(X)$ explain how to define an action of $G$ on $X$ and prove that it satisfies the necessary axioms. 

(4 marks)

(ii) Let $H < G$ be a subgroup. For $g \in G$ and $x \in G$ define $g \ast (xH) = (gx)H$. 

(a) Show that this defines a group action of $G$ on the set $G/H$ of (left) cosets of $H$. 

(4 marks)

(b) Define the homomorphism $\phi : G \to S(G/H)$ corresponding to the action in (a) and prove that 

$$\ker(\phi) = \bigcap_{x \in G} xHx^{-1}.$$ 

(5 marks)

(iii) (a) Draw a 2-dimensional shape whose symmetry group is $D_3 \cong S_3$. 

(2 marks)

(b) Prove that the direct symmetry group $\text{Dir}(\text{Dodec})$ of the regular dodecahedron has 60 elements by calculating the size of the orbit and stabilizer of a chosen face. 

(3 marks)

(c) Arguing geometrically determine how many conjugacy classes there are in $\text{Dir}(\text{Dodec})$ of elements of order 5. [You may assume that all elements of order 5 are given by rotations about axes through centres of faces.] 

(4 marks)

4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. 

(5 marks)

(ii) (a) Give the definition of a simple group. 

(2 marks)

(b) Show that there is no simple group of order 224 by considering an appropriate group action. 

(4 marks)

(c) By considering the order of elements show that a group of order $p^2q$ with $p, q$ distinct primes cannot be simple if there are $p^2$ Sylow $q$-subgroups. 

(5 marks)

(iii) Determine the number of Sylow 3-subgroups of $A_5$. 

(4 marks)

(iv) Let $G$ be a group of order $p^2q^2$ for prime $p < q$. Prove that if $G$ is not of order 36 then $G$ has a normal $q$-Sylow subgroup. 

(5 marks)

End of Question Paper