SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester
2014–2015

Time Series

Marks will be awarded for your best three answers.

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 60 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student
The plot above shows time series data consisting of 60 observations of the thickness of nitride layers (unknown units); the data was part of a larger experiment on the manufacturing of a microelectronic device.

(a) Describe the data by commenting on their structure, their variation and dynamics. (2 marks)

(b) Based on your answer in (a) or otherwise, suggest suitable time series model(s) that may be appropriate for this data. (2 marks)

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(ii) A model is to be fitted to a time series of length 100. Values of the sample autocorrelation function (ACF) and sample partial ACF (PACF) are tabulated below.

<table>
<thead>
<tr>
<th>Lag ($h$)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>ACF ($r_h$)</td>
<td>0.6</td>
<td>0.4</td>
<td>0.1</td>
<td>0.05</td>
<td>0.01</td>
</tr>
<tr>
<td>PACF ($a_h^{(h)}$)</td>
<td>⋆</td>
<td>⋆⋆</td>
<td>0.02</td>
<td>0.01</td>
<td>-0.02</td>
</tr>
</tbody>
</table>

(a) Find the omitted values (⋆ and ⋆⋆). (4 marks)

(b) Check whether the time series is stationary. (1 mark)

(c) Test whether the time series is consistent with white noise. (2 marks)

(d) Test whether the time series is consistent with moving average models. (4 marks)

(e) Test whether the time series is consistent with autoregressive models. (3 marks)

(f) Based on your answer in (c)-(e) above, suggest a time series model that may be suitable to model the data. (2 marks)

2  Consider the time series model

$$y_t = -\frac{1}{3}y_{t-1} + \epsilon_t - \frac{1}{2}\epsilon_{t-1} + \frac{1}{4}\epsilon_{t-2},$$

where $\epsilon_t$ is white noise with variance 1.

(i) Write the above model using the backward shift operator $B$. (2 marks)

(ii) Show that this model is causal and invertible. (3 marks)

(iii) Find the mean and the variance of $y_t$. (6 marks)

(iv) Find the autocorrelation function of $y_t$. (9 marks)
(i) In the context of maximum likelihood estimation of ARMA models describe briefly what is meant by conditional least squares estimation.  \(2\) marks

(ii) Consider that \(y_t\) is generated by an autoregressive model of order 2 (AR(2))

\[ y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \epsilon_t, \]

where \(\alpha_1, \alpha_2\) are the AR coefficients and \(\epsilon_t\) is white noise with variance \(\sigma^2\).

(a) Write down the conditional likelihood and the conditional log-likelihood functions of the parameters \(\alpha_1, \alpha_2\) and \(\sigma^2\), based on a collection of observations \(y_{1:n} = (y_1, y_2, \ldots, y_n)\).  \(4\) marks

(b) Using conditional least squares, show that the maximum likelihood estimates of \(\alpha_1, \alpha_2\) and \(\sigma^2\) are

\[
\hat{\alpha}_1 = \frac{\sum_{t=3}^{n} y_{t-2}^2 \sum_{t=3}^{n} y_t y_{t-1} - \sum_{t=3}^{n} y_{t-1} y_{t-2} \sum_{t=3}^{n} y_t y_{t-2}}{\sum_{t=3}^{n} y_{t-1}^2 \sum_{t=3}^{n} y_{t-2}^2 - (\sum_{t=3}^{n} y_{t-1} y_{t-2})^2},
\]

\[
\hat{\alpha}_2 = \frac{\sum_{t=3}^{n} y_{t-1}^2 \sum_{t=3}^{n} y_t y_{t-2} - \sum_{t=3}^{n} y_t y_{t-1} \sum_{t=3}^{n} y_{t-1} y_{t-2}}{\sum_{t=3}^{n} y_{t-2}^2 \sum_{t=3}^{n} y_{t-1}^2 - (\sum_{t=3}^{n} y_{t-1} y_{t-2})^2},
\]

\[
\hat{\sigma}^2 = \frac{1}{n - 2} \sum_{t=3}^{n} (y_t - \hat{\alpha}_1 y_{t-1} - \hat{\alpha}_2 y_{t-2})^2.
\]

\(14\) marks
A company trades 10 products, with the ith product projected to give a return $r_{it}$ at time $t$, for $i = 1, 2, \ldots, 10$. The company believes that each of these returns $r_{it}$ follows an autoregressive process

$$r_{it} = 0.9r_{i,t-1} + \zeta_{it},$$

where $\zeta_{it}$ is a white noise with variance 1, $\zeta_{it} \sim N(0, 1)$, and $\zeta_{it}$ is independent of $\zeta_{jt}$, for any $i \neq j$.

Due to a data recording error $r_{it}$ is not available. However, the aggregate return can be observed subject to additive noise, according to the model

$$y_t = \sum_{i=1}^{10} r_{it} + \epsilon_t,$$

where $\epsilon_t$ is a white noise with variance 1, $\epsilon_t \sim N(0, 1)$, and it is assumed that $\epsilon_t$ is independent of $\zeta_{it}$, for any $t$ and for any $i$.

(i) Define the state

$$\beta_t = \sum_{i=1}^{10} r_{it}.$$

Show that $y_t$ follows a state space model

$$y_t = x \beta_t + \epsilon_t$$

$$\beta_t = F \beta_{t-1} + \zeta_t$$

and determine $x$, $F$, $\zeta_t$ and the variance of $\zeta_t$. \hspace{1cm} (4 marks)

(ii) A prior distribution for $\beta_0$ is set as

$$\beta_0 \sim N(0, 100).$$

If the first observation is $y_1 = 2$, perform the Kalman filter iteration for $t = 1$ and obtain the posterior distribution of

$$\beta_1 \mid \{y_1 = 2\}.$$ 

(8 marks)

(iii) Using the result in (ii) obtain a 95% predictive interval for $y_2$. \hspace{1cm} (4 marks)

(iv) Describe briefly what is the likely effect on the posterior distribution of $\beta_t$ (for large $t$), if the prior distribution of $\beta_0$ changes from (a) $\beta_0 \sim N(0, 1)$ to (b) $\beta_0 \sim N(0, 1000)$. \hspace{1cm} (4 marks)

End of Question Paper