The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2014–2015**

Optics and Symplectic Geometry **2 hours 30 minutes**

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Throughout the paper I denotes an identity matrix and J denotes a matrix of the form $\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. All matrices have real entries. The standard symplectic form Ω on $\mathbb{R}^{2n}$ is defined by $\Omega(Z, Z') = Q \cdot P' - P \cdot Q'$, where $Z = (Q, P)$ and $Z' = (Q', P')$ are elements of $\mathbb{R}^{2n}$.

1 Figure 1 shows the refraction of a light ray across a parabolic boundary. The light ray enters from the left, making an angle φ with the positive z-direction, and makes an angle of θ with the normal to the boundary at the point where it crosses. The refracted ray makes an angle φ' (not shown) with the positive z-direction, and an angle θ' with the normal.

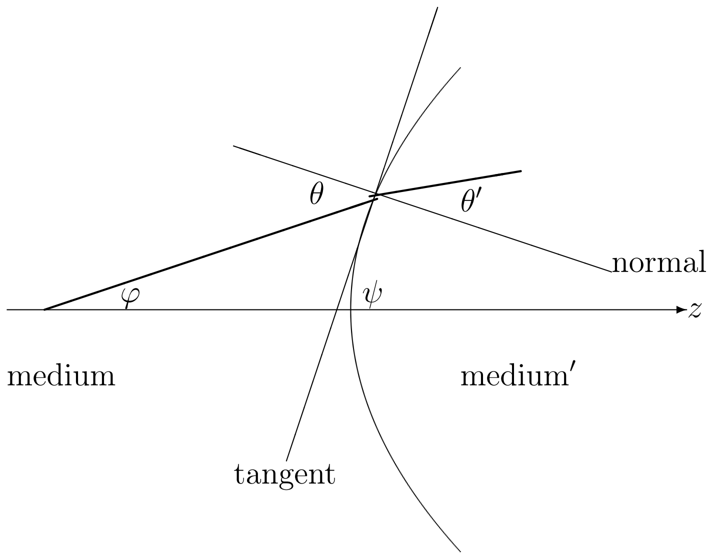

Figure 1: The angles are larger than in the assumptions for Gaussian optics, so that the diagram is clear. Here ψ is the angle between the tangent and the positive z direction. The equation of the parabolic boundary is $z = \frac{1}{2}kq^2 + z_0$ where $k, z_0 \in \mathbb{R}$ are constants, and q is measured upwards at right-angles to the z-axis. The index of refraction for the medium on the left is n and for the medium on the right is n'.

(a) List three of the assumptions made in Gaussian optics. **(3 marks)**

(b) Show, referring to Figure 1 and using additional diagrams if you wish, that

$$\varphi = \psi + \theta - \tfrac{\pi}{2} \qquad \text{and} \qquad \varphi' = \psi + \theta' - \tfrac{\pi}{2}.$$

(8 marks)

(c) Define p and p' as in the course and, using the assumptions of Gaussian optics, find the matrix $S = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that the incoming and outgoing rays are given by

$$\begin{bmatrix} q' \\ p' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}.$$

Your answer should be in terms of n, n', k.

State clearly any assumption or Law that you use which you did not give in answer to (a). You do not need to repeat the proof in (b). **(14 marks)**

2 Let $\widetilde{\mathscr{L}_3}$ denote the set of all oriented straight lines in $\mathbb{R}^3$.

(a) Define, using sketch diagrams if you wish, a bijective map from $\widetilde{\mathscr{L}_3}$ to M where

$$M := \{(X, Y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid |X|^2 = 1,\ X \boldsymbol{.} Y = 0\}.$$

Describe carefully the map $M \to \widetilde{\mathscr{L}_3}$ which is inverse to your map.

(9 marks)

(b) Fix a point $(X, Y) \in M$ and consider a smooth curve in M given by $(X(t), Y(t))$ for $t \in \mathbb{R}$ with $X(0) = X,\ Y(0) = Y$.

Writing $Q = \dot{X}(0),\ P = \dot{Y}(0)$, show that

$$Q \boldsymbol{.} X = 0, \qquad Q \boldsymbol{.} Y + X \boldsymbol{.} P = 0. \tag{$*$}$$

(2 marks)

(c) Writing

$$T_{(X,Y)} = \{(Q, P) \mid Q \boldsymbol{.} X = 0,\ Q \boldsymbol{.} Y + X \boldsymbol{.} P = 0\},$$

show that $T_{(X,Y)}$ is a symplectic subspace of $\mathbb{R}^6$ with respect to Ω.

(14 marks)

3 (a) Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ matrix in block form, where A, B, C and D denote $n \times n$ matrices.

Prove that S is symplectic if and only if the three equations

$$A^{\mathsf{T}}C = C^{\mathsf{T}}A, \qquad B^{\mathsf{T}}D = D^{\mathsf{T}}B, \qquad A^{\mathsf{T}}D - C^{\mathsf{T}}B = I,$$

hold. **(3 marks)**

(b) Now consider another $2n \times 2n$ matrix $H = \left[\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right]$ in block form, where α, β, γ and δ denote $n \times n$ matrices.

The matrix H is a *Hamiltonian matrix* if $H^{\mathsf{T}}J + JH = 0$.

Prove that H is a Hamiltonian matrix if and only if

$$\beta^{\mathsf{T}} = \beta, \qquad \gamma^{\mathsf{T}} = \gamma, \qquad \alpha^{\mathsf{T}} + \delta = 0.$$

(4 marks)

(c) Let H be a Hamiltonian $2n \times 2n$ matrix such that $I - H$ is invertible. Show that

$$S = (I + H)(I - H)^{-1}$$

is a symplectic matrix. (It is not necessary to use (a) or (b), though you may do so if you wish.) **(12 marks)**

(d) Show that a $2n \times 2n$ matrix $S = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$ is both symplectic and orthogonal if and only if $A^{\mathsf{T}}A + B^{\mathsf{T}}B = I_n$ and $A^{\mathsf{T}}B = B^{\mathsf{T}}A$. (Here I_n is the $n \times n$ identity matrix.) **(6 marks)**

4 In this question you may use the equations stated in Q3(a) without proof.

(i) Let $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ be a $2n \times 2n$ symplectic matrix in block form, where A, B, C and D denote $n \times n$ matrices.

(a) Determine, in terms of A, B, C, D, when S maps the subspace $\mathbb{R}^n \times 0$ to itself.

(b) Determine, in terms of A, B, C, D, when S maps the subspace $\mathbb{R}^n \times 0$ to itself *and* the subspace $0 \times \mathbb{R}^n$ to itself.

(c) Of the symplectic matrices found in (b), determine those which also map the subspace $\{(Q, Q) \mid Q \in \mathbb{R}^n\}$ to itself, expressing your answer in terms of conditions on A.

(6 marks)

(ii) Let $\{G_1, \ldots, G_n\}$ be a basis for an n–dimensional subspace L of $\mathbb{R}^{2n}$. Write the G_i as the columns of a matrix and write the matrix in block form as

$$\begin{bmatrix} M \\ N \end{bmatrix}$$

where M and N are $n \times n$ matrices.

(a) Prove that L is Lagrangian if and only if $M^\top N$ is symmetric. Determine when L is transverse to $\mathbb{R}^n \times 0$. **(8 marks)**

(b) Let L be a Lagrangian subspace of $\mathbb{R}^{2n}$ transverse to $\mathbb{R}^n \times 0$ and corresponding to M and N as above.

Find A, B, C, D so that the symplectic matrix $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ maps $\mathbb{R}^n \times 0$ to itself and $0 \times \mathbb{R}^n$ to L.

Determine all the symplectic matrices S which map $\mathbb{R}^n \times 0$ to itself and $0 \times \mathbb{R}^n$ to L. **(11 marks)**

5 Let $\Gamma\colon \mathbb{R}^4 \to \mathbb{R}$ be a smooth map and write $\Gamma = \Gamma(Q, Q')$ where $Q, Q' \in \mathbb{R}^2$.

(a) Explain what is meant by $\dfrac{\partial \Gamma}{\partial Q}$ and $\dfrac{\partial \Gamma}{\partial Q'}$, and give their expressions in components. **(3 marks)**

(b) Write $H = \dfrac{\partial \Gamma}{\partial Q}$, $K = \dfrac{\partial \Gamma}{\partial Q'}$. Show that

$$\frac{\partial H}{\partial Q'} = \left(\frac{\partial K}{\partial Q}\right)^{\mathsf{T}}.$$

and that $\dfrac{\partial H}{\partial Q}$ and $\dfrac{\partial K}{\partial Q'}$ are symmetric. **(5 marks)**

(c) Now assume that

$$\det \frac{\partial H}{\partial Q'} = \det \frac{\partial K}{\partial Q} \neq 0, \qquad (*)$$

and write $P = H(Q, Q')$. Condition $(*)$ ensures that $P = H(Q, Q')$ can be solved for Q' in terms of P and Q. That is, there exists $F\colon \mathbb{R}^4 \to \mathbb{R}^2$, defined on an open set in $\mathbb{R}^4$, such that $Q' = F(Q, P)$ if and only if $P = H(Q, Q')$. Show that

$$\frac{\partial H}{\partial Q'}\frac{\partial F}{\partial P} = I. \qquad \text{and} \qquad \frac{\partial H}{\partial Q} + \frac{\partial H}{\partial Q'}\frac{\partial F}{\partial Q} = 0.$$

(8 marks)

(d) Now define $G(Q, P) = -K(Q, F(Q, P))$. You may assume that

$$\frac{\partial G}{\partial Q} = -\frac{\partial K}{\partial Q} - \frac{\partial K}{\partial Q'}\frac{\partial F}{\partial Q} \qquad \text{and} \qquad \frac{\partial G}{\partial P} = -\frac{\partial K}{\partial Q'}\frac{\partial F}{\partial P}.$$

Prove that $\left(\dfrac{\partial F}{\partial Q}\right)^{\mathsf{T}} \dfrac{\partial G}{\partial Q}$ is a symmetric matrix. **(9 marks)**

End of Question Paper