1. (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic.

   \( \frac{\partial u}{\partial p} + 3 \frac{\partial^2 u}{\partial x^2} - 2 \frac{\partial^2 u}{\partial p^2} = u \)  
   (1 mark)

   \( \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + 3 = -\frac{\partial u^2}{\partial y^2} - 2u \)  
   (1 mark)

   \( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial x} = 5 \)  
   (1 mark)

(ii) Discuss the benefits/drawbacks of the three systems of numerical solutions, i.e. implicit, explicit and Crank-Nicolson.  
   (6 marks)

(iii) Consider the following differential equation

\[
3 \frac{\partial^2 u}{\partial y^2} + 2 \frac{\partial^2 u}{\partial x^2} + 42u = 0,
\]

By using the explicit difference scheme, solve the differential equation between \( 0 \leq x \leq 1, 0 \leq y \leq 1 \) for

\[
\Delta x = h = \frac{1}{3}, \quad \Delta y = k = \frac{1}{2},
\]

and the following boundary conditions

\[
u(1, y) = 0, \quad u(x, 1) = 1 - x^2,
\]

under the assumption that the solution is symmetric across both the \( x \) and \( y \) axes, i.e. that \( u(x, y) = u(x, -y) = u(-x, y) \).  
   (15 marks)

(iv) State the order of the error.  
   (1 mark)
2  (i) Discuss the advantages/disadvantages of LU factorisation over ‘traditional’ matrix inversion.  
   \( (2 \text{ marks}) \)

   (ii) Determine the L and U matrices for the following system

   \[
   Ax = b, \quad A = \begin{bmatrix}
   8 & 2 & 9 \\
   4 & 9 & 4 \\
   6 & 7 & 9 
\end{bmatrix}, \quad b = \begin{bmatrix}
   4 \\
   5 \\
   3 
\end{bmatrix}
   \]

   \( (5 \text{ marks}) \)

   (iii) Determine \( L^{-1} \) and \( U^{-1} \) \( (4 \text{ marks}) \)

   (iv) Hence find the column vector \( x \) \( (3 \text{ marks}) \)

   (v) Find the LU decomposition for the following tri-diagonal matrix

   \[
   M = \begin{bmatrix}
   3 & 2 & 0 & 0 & 0 \\
   1 & 3 & 2 & 0 & 0 \\
   0 & 1 & 3 & 2 & 0 \\
   0 & 0 & 1 & 3 & 2 \\
   0 & 0 & 0 & 1 & 3 
\end{bmatrix}
   \]

   \( (6 \text{ marks}) \)

   (vi) Using your results from part (v), or otherwise, show \( M^{-1} \) to be

   \[
   M^{-1} = \frac{1}{63} \begin{bmatrix}
   31 & -30 & 28 & -24 & 16 \\
   -15 & 45 & -42 & 36 & -24 \\
   7 & -21 & 49 & -42 & 28 \\
   -3 & 9 & -21 & 45 & -30 \\
   1 & -3 & 7 & -15 & 31 
\end{bmatrix}
   \]

   \( (5 \text{ marks}) \)

3  (i) Write down the three properties of cubic splines and why they are important.  
   \( (3 \text{ marks}) \)

   (ii) Write down the values for \( \sigma_0 \) and \( \sigma_n \) under the assumption of

   \[
   f''(x_0) = 0, \quad f'(x_n) = 0.
   \]

   \( (2 \text{ marks}) \)

   (iii) Using the conditions derived in (ii), determine the cubic spline between the following data points

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>\pi/3</th>
<th>2\pi/3</th>
<th>\pi</th>
</tr>
</thead>
<tbody>
<tr>
<td>f(x)</td>
<td>1</td>
<td>1/2</td>
<td>-1/2</td>
<td>-1</td>
</tr>
</tbody>
</table>

   \( (18 \text{ marks}) \)
The set \((1, 0, 0) = a_1, (0, 1, 0) = a_2, (1, 1, 1) = a_3\) is in \(\mathbb{R}^3\).

(i) Show that this set is independent but not orthogonal.  

(ii) Find \(c_i, d_i, i = 1, 2, 3\), such that

\[
f = (1, 2, 4) = \sum_{i=1}^{3} c_i a_i,
\]

\[
g = (-1, -1, 2) = \sum_{i=1}^{3} d_i a_i.
\]

(iii) Show that

\[
||f||^2 \neq \sum_{i=1}^{3} |c_i|^2,
\]

\[
||g||^2 \neq \sum_{i=1}^{3} |d_i|^2,
\]

\[
(f, g) \neq \sum_{i=1}^{3} c_i d_i.
\]

(iv) Explain briefly why the equality does not hold in part (iii) and thus the advantages of using an orthonormal basis.
5 (i) Digital signals \( \{f[n]\} \) \((n = 1, 2, 3)\) of length 3 are obtained by sampling a random signal \( f(t) \) at intervals \( T/2 \), where \( f(t) \) has autocorrelation function

\[
R_f(\tau) = \sigma^2(Sa(\pi \tau/T))^2
\]

where \( Sa(x) = \sin(x)/x \), and \( \sigma, \tau \) and \( T \) are constant.

Write down the correlation matrix \( R \), and use it to derive the K-L basis.

(20 marks)

(ii) Such signals are to be compressed using only two members of the K-L basis. Find them. What is the associated mean square error?

(5 marks)

6 (i) In \( L^2[0, 1] \) find constants \( a, b, c, d, e, f \) so that the vectors (i.e. functions)

\[
\phi_1(t) = a, \quad \phi_2(t) = b + ct, \quad \phi_3(t) = d + et + ft^2
\]

form an orthonormal set.

(17 marks)

(ii) Find the best approximation to the vector \( f(t) = t^3 \) using \( \{\phi_i\}_{i=1}^3 \).

(8 marks)

End of Question Paper
Formulae Sheet

Notation:
\[ U(x_i, t_j) \equiv U_{ij} \]

Forward difference formula for \( \frac{\partial U}{\partial t} \):
\[ \frac{\partial U}{\partial t} \approx \frac{U_{ij+1} - U_{ij}}{\Delta t} \]

Backward difference formula for \( \frac{\partial U}{\partial t} \):
\[ \frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{ij-1}}{\Delta t} \]

Central difference formula for \( \frac{\partial U}{\partial x} \):
\[ \frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} \]

Central difference formula for \( \frac{\partial^2 U}{\partial x^2} \):
\[ \frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2} \]