Vectors and Mechanics

Attempt all the questions. The allocation of marks is shown in brackets. The total number of marks available is 60.

1. Let \( OABC \) be a parallelogram where the midpoints of the lines \( OA, AB \) and \( BC \) are \( L, M \) and \( N \), respectively. Let \( \mathbf{a} = \overrightarrow{OA} \) and \( \mathbf{c} = \overrightarrow{OC} \).

   (i) Express \( \overrightarrow{OL}, \overrightarrow{OM} \) and \( \overrightarrow{ON} \) in terms of \( \mathbf{a} \) and \( \mathbf{c} \).

   (ii) Express \( \overrightarrow{LM} \cdot \overrightarrow{MN} \) in terms of \( |\mathbf{a}| \) and \( |\mathbf{c}| \). \( (3 \text{ marks}) \)

2. Relative to the origin \( O \), points \( A \) and \( B \) have position vectors \( \mathbf{a} = 2\mathbf{i} - \mathbf{j} \) and \( \mathbf{b} = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \), respectively.

   Find

   (i) the angle in radians between \( \mathbf{a} \) and \( \mathbf{b} \) correct to two decimal places;

   (ii) the component of \( \mathbf{b} \) along \( \mathbf{a} \);

   (iii) \( \mathbf{a} \times \mathbf{b} \);

   (iv) the area of triangle \( AOB \) correct to two decimal places. \( (6 \text{ marks}) \)

3. Simplify the following expressions

   (i) \( \mathbf{i} + (\mathbf{j} \times \mathbf{k}) \);

   (ii) \( (\mathbf{i} \times \mathbf{k}) \cdot \mathbf{j} \). \( (2 \text{ marks}) \)

4. (i) Find a parametric vector equation of line \( L_1 \) which passes through a point with position vector \( 3\mathbf{i} - \mathbf{j} - 3\mathbf{k} \) and is in the direction of vector \( -\mathbf{i} + 2\mathbf{j} + 6\mathbf{k} \).

   (ii) Find a parametric vector equation of line \( L_2 \) which passes through points with position vectors \( \mathbf{i} + 2\mathbf{j} + \mathbf{k} \) and \( 6\mathbf{i} - 3\mathbf{j} + 11\mathbf{k} \).

   (iii) Find out if \( L_1 \) and \( L_2 \) intersect and, if so, state their point of intersection. \( (6 \text{ marks}) \)
A point $P$ with the position vector $\mathbf{p}$ lies on a line $L$ with the parametric vector equation $\mathbf{r} = \mathbf{a} + \lambda \mathbf{c}$ such that $\mathbf{p} \cdot \mathbf{c} = 0$.

(i) Find $\lambda$ for the position vector $\mathbf{p}$.

(ii) Assume that the vector equation of line $L$ can also be written in the form $\mathbf{r} \times \mathbf{c} = \mathbf{d}$, where $\mathbf{d}$ is a constant vector. Hence show that $\mathbf{p} = \frac{1}{|\mathbf{c}|^2}(\mathbf{c} \times \mathbf{d})$.

(3 marks)

A projectile is launched from the origin $O$ with speed $V$ at an angle $\theta$ above the horizontal. Ignore the effect of air resistance.

If the horizontal and vertical displacements of the projectile at time $t$ are $x$ and $z$ respectively, write down the equations for $x$ and $z$ in terms of $V$, $\theta$ and the acceleration due to gravity $g$.

Assume an area of ground is completely flat and level with the $x$-axis. Show that a projectile launched from this ground will have a maximum range when $\theta = \pi/4$. Hence derive an expression for this maximum range in terms of $V$ and $g$.

(4 marks)

(i) The position of a thrown stone is

$$\mathbf{r}(t) = (1.6 + 12 t) \mathbf{i} + (15 t - 4.9 t^2) \mathbf{j},$$

where the units are m and s as appropriate. Find its velocity and acceleration.

(2 marks)

(ii) A car is travelling east at 72 km h$^{-1}$. It rounds a curve and 5 seconds later it is travelling North at 72 km h$^{-1}$. Find the average acceleration of the car over the 5s.

(5 marks)

(iii) John’s car has a mass of 2000 kg. It is at rest on the road because it runs out of fuel. John then pushes the car exerting a force of 300 N. How long does it take for the car to reach a velocity of 3 m s$^{-1}$?

(4 marks)

(i) One end of a light, taut inextensible string of length $L$ is attached to a fixed point $O$ and a particle $P$ is attached to the other end of the string. The particle moves on a smooth track banked at an angle $\alpha$ to the horizontal. The particle moves in a horizontal circle with $O$ at its centre, so that the string is horizontal. Draw a clear force diagram showing all the forces acting on the particle.

(3 marks)

(ii) Two small beads are threaded on a vertical smooth circular wire. The beads are connected by a light, taut inextensible string which runs round the wire. Draw a clear diagram showing the forces on the two beads.

(4 marks)

(iii) An elastic string of length 1.5 m is stretched to 2 m. If its modulus of elasticity is 12 N, find the tension in the string.

(3 marks)
9 (i) A block of mass 4 kg lies on a rough plane banked at an angle of 30° to the horizontal. The coefficient of friction between the block and the plane is \( \frac{3}{4} \). Show that in the absence of any external force the block does not slip down the plane if it is initially at rest. \( (3 \text{ marks}) \)

(ii) What is the smallest force parallel to the plane which can be applied to make the block just begin to move down the plane. \( (2 \text{ marks}) \)

(iii) What is the smallest force parallel to the plane which can be applied to make the block just begin to move up the plane. \( (2 \text{ marks}) \)

Note: consider acceleration due to gravity to be \( g = 10 \text{ m s}^{-2} \) in Q. 9

10 For each of the following physical quantities, state whether they are vectors or scalars and give their dimensions: (a) Momentum; (b) Power; (c) Stiffness. \( (3 \text{ marks}) \)

11 A penny of mass 0.1 kg is placed on a horizontal turntable that is then rotated at a fixed rate of 90 revolutions per minute. The penny is placed on the table at a distance \( r \) from the axis of rotation. If the maximum frictional force between the turntable and the penny is 0.6 N, calculate the maximum distance \( r \) at which the penny would stay on the turntable at this rotation rate. \( (5 \text{ marks}) \)

End of Question Paper