**Data provided: Formula Sheet**



The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2015–2016**

**MAS140 Mathematics (Chemical)** **3 hours**

*Attempt* ***ALL*** *questions.*

*Each question in Section A carries 3 marks, each question in Section B carries 8 marks.*

*All solutions should be justified in full. Calculators should be relied upon only for simple steps like basic arithmetic and plugging numbers into elementary functions.*

## Section A

**A1** Let $f(x) = \dfrac{x}{x+4}$. Sketch the curve $y = f(x)$.

**A2** Let $f(x) = e^{2x} - 1$. Find $f^{-1}(x)$ and state its domain and range.

**A3** If $f(x, y) = 4x^2\sqrt{y} + 5\cos(xy)$, find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$.

**A4** Evaluate $\displaystyle\lim_{x\to 0}\left(\frac{x\tanh x}{\sin 2x}\right)$ using l'Hôpital's Rule.

**A5** Find all the complex numbers $z$ for which $|z - 1 - i| = 1$ and $\mathrm{Re}(z) = \mathrm{Im}(z)$.

**A6** If $\mathbf{a} = (7, -2, -5)$ and $\mathbf{b} = (5, 1, 3)$, evaluate $\mathbf{a} \cdot \mathbf{b}$. Find a non-zero vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

**A7** Find the definite integral $\int_0^\pi (x+1) \sin \frac{x}{2}\, dx$.

**A8** Find the indefinite integral $\int x(\cos x^2)^2\, dx$.

**A9** Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$. Find $A^T$ and $B^T$ and hence show that $(AB)^T = B^T A^T$ .

**A10** Find the general solution of the differential equation

$$x\frac{dy}{dx} = \frac{e^x}{x^2} - 3y.$$

**A11** Find $A^{-1}$ for the matrix $A = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$. Use this to solve the simultaneous equations

$$\begin{aligned} 2x + 4y &= 14 \\ x - 3y &= -8. \end{aligned}$$

**A12** Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$$

such that $y = 0$ when $x = 0$.

## Section B

**B1** Find all the stationary points of $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 - 1$ and show that the stationary points not at $(0, 0)$ are minima.

**B2** By evaluating all the necessary derivatives of $y = \ln(1 + x^2)$, find the first 2 non-zero terms of the Maclaurin Series expansion of $y$. Show that this series can also be obtained from the Maclaurin Series of $y = \ln(1+x)$ given on the Formula Sheet.

**B3** Find the modulus and principal argument of the complex numbers $z_1 = 1 + i$ and $z_2 = \sqrt{3} + i$. Hence find all complex numbers $z$ that satisfy the equation $z^6 = \frac{z_1}{z_2}$ and plot them on an Argand diagram.

**B4** The position vector of a particle, $\mathbf{r}(t)$, is given by

$$\mathbf{r}(t) = (2t^2, t^2 - 4t, 3t - 5).$$

(i) Find the velocity and acceleration vectors of the particle.

(ii) Find the unit vector in the direction of $\mathbf{r}$ at $t = 1$. Hence show that, in this direction, the component of the velocity is 4 times the component of the acceleration.

**B5** Evaluate the definite integral

$$\int_1^2 \frac{2t^2 + 3t + 1}{t^3 + t}\, dt$$

writing your answer to 2 decimal places.

**B6** Find the value of $\alpha$ for which the following system of equations has infinitely many solutions and then find those solutions.

$$\begin{aligned} 4x - y - z &= 2 \\ 2x + \alpha y + z &= 4 \\ x - 2y - 2z &= -3 \end{aligned}$$

For the case $\alpha = -2$, without solving the equations state how many solutions you would expect and write down the solution of the corresponding homogeneous system of equations.

**B7** Let $A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$. Find all eigenvalues and eigenvectors of $A$.

**B8** Using Laplace Transforms or otherwise, find the solution to the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$$

subject to the initial conditions $y = 0$ and $\frac{dy}{dt} = 0$ at $t = 0$.

**End of Question Paper**

# MAS140/151/152/156 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

## Trigonometry

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \mathrm{cosec}^2 x$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$2\sin x \cos x = \sin 2x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$\tan(x \pm y) = \dfrac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

$a\cos x + b\sin x = R\cos(x - \alpha)$

where $R = \sqrt{a^2 + b^2}$ ,

$\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$

$2\cos x \cos y = \cos(x+y) + \cos(x-y)$

$2\sin x \sin y = \cos(x-y) - \cos(x+y)$

$2\sin x \cos y = \sin(x+y) + \sin(x-y)$

$\cos x = \dfrac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$

$\sin x = \dfrac{2\tan(x/2)}{1 + \tan^2(x/2)}$

$\tan x = \dfrac{2\tan(x/2)}{1 - \tan^2(x/2)}$

## Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$\tanh x = \dfrac{\sinh x}{\cosh x}$

$\coth x = \dfrac{\cosh x}{\sinh x}$

$\mathrm{sech} x = \dfrac{1}{\cosh x}$

$\cosh^2 x - \sinh^2 x = 1$

$2\cosh^2 x = 1 + \cosh 2x$

$2\sinh^2 x = \cosh 2x - 1$

$2\sinh x \cosh x = \sinh 2x$

$\mathrm{sech}^2 x = 1 - \tanh^2 x$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, all $x$

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$

$\tanh^{-1} x = \dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right)$, $|x| < 1$

Series

Sum of an arithmetic series:

$$\frac{\text{first term} \ + \ \text{last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r}x^r + \ldots$

where $\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

If $n$ is a positive integer then the series terminates and the result is true for all $x$, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \leq 1)$$

Differentiation

| Function | Derivative | Function | Derivative |
|---|---|---|---|
| $\sin x$ | $\cos x$ | $\cos x$ | $-\sin x$ |
| $\tan x$ | $\sec^2 x$ | $\cot x$ | $-\mathrm{cosec}^2 x$ |
| $\sin^{-1} x$ | $\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$ | $\cos^{-1} x$ | $-\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$ |
| $\tan^{-1} x$ | $\dfrac{1}{1+x^2}$ | $\cot^{-1} x$ | $-\dfrac{1}{1+x^2}$ |
| $\sinh x$ | $\cosh x$ | $\cosh x$ | $\sinh x$ |
| $\tanh x$ | $\dfrac{1}{\cosh^2 x}$ | $\coth x$ | $-\dfrac{1}{\sinh^2 x}$ |
| $\sinh^{-1} x$ | $\dfrac{1}{\sqrt{x^2+1}}$ | $\cosh^{-1} x$ | $\dfrac{1}{\sqrt{x^2-1}}\,,\ \lvert x\rvert > 1$ |
| $\tanh^{-1} x$ | $\dfrac{1}{1-x^2}\,,\ \lvert x\rvert < 1$ | | |
| $\coth^{-1} x$ | $\dfrac{1}{1-x^2}\,,\ \lvert x\rvert > 1$ | | |

Integration

In the following table the constants of integration have been omitted.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \quad (a > 0,\ a \neq 1)$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \mathrm{cosec}^2 x\,dx = -\cot x$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \mathrm{sech}^2 x\,dx = \tanh x$$

$$\int \mathrm{cosech}^2 x\,dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad \left(= \tanh^{-1}\frac{x}{a} \quad \text{if } |x| < a\right)$$

$$\int \mathrm{cosec}\,x\,dx = \ln\tan\left(\frac{x}{2}\right) \quad \text{or} \quad \ln(\mathrm{cosec}\,x - \cot x)$$

$$\int \sec x\,dx = \ln\tan\left(\frac{x}{2} + \frac{\pi}{4}\right) \quad \text{or} \quad \ln(\sec x + \tan x)$$

$$\int \mathrm{cosech}\,x\,dx = \ln\tanh\left(\frac{x}{2}\right)$$

## Integration by parts

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

## Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha,\, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt$$

## Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

## Table of Laplace transforms

| Function $f(t)$ | Laplace transform $F(s)$ |
|---|---|
| $t^n$ | $\dfrac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \ldots$) |
| $e^{at}$ | $\dfrac{1}{s-a}$ |
| $\sin \omega t$ | $\dfrac{\omega}{s^2+\omega^2}$ |
| $\cos \omega t$ | $\dfrac{s}{s^2+\omega^2}$ |
| $\sinh \omega t$ | $\dfrac{\omega}{s^2-\omega^2}$ |
| $\cosh \omega t$ | $\dfrac{s}{s^2-\omega^2}$ |
| $e^{at} f(t)$ | $F(s-a)$ (shift theorem) |
| $f'(t)$ | $sF(s) - f(0)$ |
| $f''(t)$ | $s^2F(s) - sf(0) - f'(0)$ |