



The  
University  
Of  
Sheffield.

**MAS156**

**SCHOOL OF MATHEMATICS AND STATISTICS**

**Spring Semester  
2015–2016**

**MAS156 Mathematics (Electrical and Aerospace)**

**3 hours**

Attempt **ALL** questions.

Each question in Section A carries 3 marks,  
each question in Section B carries 8 marks.

All solutions should be justified in full. Calculators should be relied upon only for simple steps like basic arithmetic and plugging numbers into elementary functions.

### Section A

**A1** Let  $f(x) = \frac{x}{x+4}$ . Sketch the curve  $y = f(x)$ .

**A2** Let  $f(x) = e^{2x} - 1$ . Find  $f^{-1}(x)$  and state its domain and range.

**A3** If  $f(x, y) = 4x^2\sqrt{y} + 5\cos(xy)$ , find  $\frac{\partial f}{\partial x}$ ,  $\frac{\partial f}{\partial y}$ .

**A4** Evaluate  $\lim_{x \rightarrow 0} \left( \frac{x \tanh x}{\sin 2x} \right)$  using l'Hôpital's Rule.

**A5** Find all the complex numbers  $z$  for which  $|z - 1 - i| = 1$  and  $\operatorname{Re}(z) = \operatorname{Im}(z)$ .

**A6** If  $\mathbf{a} = (7, -2, -5)$  and  $\mathbf{b} = (5, 1, 3)$ , evaluate  $\mathbf{a} \cdot \mathbf{b}$ . Find a non-zero vector perpendicular to both  $\mathbf{a}$  and  $\mathbf{b}$ .

**A7** Find the definite integral  $\int_0^\pi (x + 1) \sin \frac{x}{2} dx$ .

**A8** Find the indefinite integral  $\int x(\cos x^2)^2 dx$ .

**A9** Let  $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$  and  $B = \begin{bmatrix} 2 & 1 \\ -1 & 3 \end{bmatrix}$ . Find  $A^T$  and  $B^T$  and hence show that  $(AB)^T = B^T A^T$ .

**A10** Find the general solution of the differential equation

$$x \frac{dy}{dx} = \frac{e^x}{x^2} - 3y.$$

**A11** Find  $A^{-1}$  for the matrix  $A = \begin{bmatrix} 2 & 4 \\ 1 & -3 \end{bmatrix}$ . Use this to solve the simultaneous equations

$$\begin{aligned} 2x + 4y &= 14 \\ x - 3y &= -8. \end{aligned}$$

**A12** Find the general solution of the equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4y = 0$$

such that  $y = 0$  when  $x = 0$ .

## Section B

**B1** Find all the stationary points of  $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2 - 1$  and show that the stationary points not at  $(0, 0)$  are minima.

**B2** By evaluating all the necessary derivatives of  $y = \ln(1 + x^2)$ , find the first 2 non-zero terms of the Maclaurin Series expansion of  $y$ . Show that this series can also be obtained from the Maclaurin Series of  $y = \ln(1 + x)$  given on the Formula Sheet.

**B3** Find the modulus and principal argument of the complex numbers  $z_1 = 1 + i$  and  $z_2 = \sqrt{3} + i$ . Hence find all complex numbers  $z$  that satisfy the equation  $z^6 = \frac{z_1}{z_2}$  and plot them on an Argand diagram.

**B4** The position vector of a particle,  $\mathbf{r}(t)$ , is given by

$$\mathbf{r}(t) = (2t^2, t^2 - 4t, 3t - 5).$$

- (i) Find the velocity and acceleration vectors of the particle.
- (ii) Find the unit vector in the direction of  $\mathbf{r}$  at  $t = 1$ . Hence show that, in this direction, the component of the velocity is 4 times the component of the acceleration.

**B5** Evaluate the definite integral

$$\int_1^2 \frac{2t^2 + 3t + 1}{t^3 + t} dt$$

writing your answer to 2 decimal places.

**B6** Find the value of  $\alpha$  for which the following system of equations has infinitely many solutions and then find those solutions.

$$\begin{aligned} 4x - y - z &= 2 \\ 2x + \alpha y + z &= 4 \\ x - 2y - 2z &= -3 \end{aligned}$$

For the case  $\alpha = -2$ , without solving the equations state how many solutions you would expect and write down the solution of the corresponding homogeneous system of equations.

**B7** Let  $A = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 1 & -2 \\ -2 & 0 & 5 \end{bmatrix}$ . Find all eigenvalues and eigenvectors of  $A$ .

**B8** Using Laplace Transforms or otherwise, find the solution to the differential equation

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$$

subject to the initial conditions  $y = 0$  and  $\frac{dy}{dt} = 0$  at  $t = 0$ .

**End of Question Paper**

MAS140/151/152/156 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$$\sin^2 x + \cos^2 x = 1$$

$$1 + \tan^2 x = \sec^2 x$$

$$1 + \cot^2 x = \operatorname{cosec}^2 x$$

$$2 \sin^2 x = 1 - \cos 2x$$

$$2 \cos^2 x = 1 + \cos 2x$$

$$2 \sin x \cos x = \sin 2x$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$$a \cos x + b \sin x = R \cos(x - \alpha)$$

$$\text{where } R = \sqrt{a^2 + b^2},$$

$$\cos \alpha = \frac{a}{R} \quad \text{and} \quad \sin \alpha = \frac{b}{R}$$

$$2 \cos x \cos y = \cos(x + y) + \cos(x - y)$$

$$2 \sin x \sin y = \cos(x - y) - \cos(x + y)$$

$$2 \sin x \cos y = \sin(x + y) + \sin(x - y)$$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2 \tan(x/2)}{1 + \tan^2(x/2)}$$

$$\tan x = \frac{2 \tan(x/2)}{1 - \tan^2(x/2)}$$

Hyperbolic Functions

$$\sinh x = \frac{1}{2}(e^x - e^{-x})$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x})$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{coth} x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$2 \cosh^2 x = 1 + \cosh 2x$$

$$2 \sinh^2 x = \cosh 2x - 1$$

$$2 \sinh x \cosh x = \sinh 2x$$

$$\operatorname{sech}^2 x = 1 - \tanh^2 x$$

$$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1}), \text{ all } x$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}), \quad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right), \quad |x| < 1$$

## Series

Sum of an arithmetic series:

$$\frac{\text{first term} + \text{last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series:  $1 + x + x^2 + \dots + x^{n-1} = \frac{1 - x^n}{1 - x}$

Binomial theorem:  $(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \dots + \binom{n}{r}x^r + \dots$

$$\text{where } \binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}$$

If  $n$  is a positive integer then the series terminates and the result is true for all  $x$ , otherwise, the series is infinite and only converges for  $|x| < 1$ .

$$\left. \begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ \sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots \\ \cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots \\ \exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \end{aligned} \right\} \text{valid for all } x$$
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \quad (-1 < x \leq 1)$$

## Differentiation

<u>Function</u>	<u>Derivative</u>	<u>Function</u>	<u>Derivative</u>
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\operatorname{cosec}^2 x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}},  x  < 1$	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}},  x  < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\cot^{-1} x$	$-\frac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\frac{1}{\cosh^2 x}$	$\coth x$	$-\frac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\frac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\frac{1}{\sqrt{x^2-1}},  x  > 1$
$\tanh^{-1} x$	$\frac{1}{1-x^2},  x  < 1$		
$\coth^{-1} x$	$\frac{1}{1-x^2},  x  > 1$		

## Integration

In the following table the constants of integration have been omitted.

$$\int x^n dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln |x|$$

$$\int e^x dx = e^x$$

$$\int a^x dx = \frac{a^x}{\ln a} \quad (a > 0, a \neq 1)$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int \sec^2 x dx = \tan x$$

$$\int \operatorname{cosec}^2 x dx = -\cot x$$

$$\int \sinh x dx = \cosh x$$

$$\int \cosh x dx = \sinh x$$

$$\int \operatorname{sech}^2 x dx = \tanh x$$

$$\int \operatorname{cosech}^2 x dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} \quad (|x| < a)$$

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 + a^2}} = \sinh^{-1} \frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad (|x| > a)$$

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln \left| \frac{a+x}{a-x} \right| \quad (= \tanh^{-1} \frac{x}{a} \quad \text{if } |x| < a)$$

$$\int \operatorname{cosec} x dx = \ln \tan \left( \frac{x}{2} \right) \quad \text{or} \quad \ln (\operatorname{cosec} x - \cot x)$$

$$\int \sec x dx = \ln \tan \left( \frac{x}{2} + \frac{\pi}{4} \right) \quad \text{or} \quad \ln (\sec x + \tan x)$$

$$\int \operatorname{cosech} x dx = \ln \tanh \left( \frac{x}{2} \right)$$

### Integration by parts

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx$$

### Variable substitution in definite integral

If  $x = \varphi(t)$  is a monotonic function in the interval  $[\alpha, \beta]$  and  $a = \varphi(\alpha)$ ,  $b = \varphi(\beta)$ , then

$$\int_a^b f(x) dx = \int_\alpha^\beta f(\varphi(t)) \varphi'(t) dt$$

### Variable substitution for a rational function of $\sin x$ and $\cos x$

Let  $t = \tan\left(\frac{x}{2}\right)$  then  $\sin x = \frac{2t}{1+t^2}$ ,  $\cos x = \frac{1-t^2}{1+t^2}$  and  $\frac{dx}{dt} = \frac{2}{1+t^2}$ .

### Table of Laplace transforms

<u>Function <math>f(t)</math></u>	<u>Laplace transform <math>F(s)</math></u>
$t^n$	$\frac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \dots$ )
$e^{at}$	$\frac{1}{s-a}$
$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$
$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$
$e^{at} f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$