SCHOOL OF MATHEMATICS AND STATISTICS

Algebra

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 60 marks.

1  (i) Let \( G \) be a group, and \( x \in G \) a fixed element. Define the \textit{conjugacy class} \( \text{con} \{ G \} (x) \) and the \textit{centraliser} \( \text{cent} \{ G \} (x) \). \hfill (2 marks)

(ii) In the permutation group \( S_5 \), let \( \alpha = (12)(345) \) and \( \beta = (15243) \). Express \( \beta \alpha \beta^{-1} \) as a product of disjoint cycles, by applying the permutation \( \beta \) to the entries in the expression for \( \alpha \). Note that you have just produced an element belonging to the conjugacy class \( \text{conj}_{S_5}(\alpha) \), and different from \( \alpha \) itself. Determine how many elements \( \text{conj}_{S_5}(\alpha) \) contains, taking care not to count any twice. By considering a permutation of the entries of \( \alpha \) that produces a different expression for the same permutation \( \alpha \), write down a non-identity element of \( \text{cent}_{S_5}(\alpha) \), different from \( \alpha \). What is the cardinality of \( \text{cent}_{S_5}(\alpha) \)? \hfill (4 marks)

2  Let \( G \) be a group. We define the \textit{centre} of \( G \) to be

\[
Z(G) := \{ g \in G \mid gx = xg \ \forall x \in G \}.
\]

Prove that \( Z(G) \) is a normal subgroup of \( G \), by applying the subgroup criterion to show that it is a subgroup, and considering left and right cosets to show that it is normal. \hfill (4 marks)

3  Consider the map \( \theta : Z \to \text{GL}_2(\mathbb{R}) \) defined by \( \theta(n) := J^n \), where \( J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

Here \( Z \) is the group of integers under addition, and \( \text{GL}_2(\mathbb{R}) \) is the group of invertible 2-by-2 real matrices, under multiplication. Prove that \( \theta \) is a group homomorphism. What does the First Isomorphism Theorem tell us in this particular example? It is not enough just to state the general theorem. \hfill (4 marks)

Turn Over
Consider the map \( \theta : \mathbb{C} \rightarrow M_2(\mathbb{R}) \) defined by \( \theta(a+bi) := \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \), where \( a, b \in \mathbb{R} \).

Prove that \( \theta \) is a ring homomorphism. Give two independent justifications for the statement that it is not a ring isomorphism between \( \mathbb{C} \) and \( M_2(\mathbb{R}) \). (You could try to show that it is not a bijection. You could also consider that any “structural” property of a ring is shared by any other ring isomorphic to it.) \( 5 \) marks

(i) Let \( R \) be a non-zero commutative ring. What does it mean to say that \( R \) is a field? \( 1 \) mark

(ii) Give a brief reason why \( \mathbb{F}_{13} := \mathbb{Z}/13\mathbb{Z} \) is a field. What is the multiplicative inverse in \( \mathbb{F}_{13} \) of \( 5 \)? \( 2 \) marks

(iii) In the quotient ring \( R := \mathbb{F}_{13}[x]/(x^2+1) \), calculate \( [x-5][x-8] \), expressing your answer in as simple a form as possible. Is \( R \) a field? Is \( \mathbb{F}_3[x]/(x^2+1) \) a field? (You might like to consider the analogy with part (ii), replacing \( \mathbb{Z} \) by \( \mathbb{F}_3[x] \) and 13 by \( x^2+1 \).) \( 4 \) marks

Consider the map \( \theta : M_2(\mathbb{R}) \rightarrow \mathbb{R} \) defined by \( \theta(A) := \det(A) \), the determinant of \( A \). Is \( \theta \) a ring homomorphism? Justify your answer. \( 2 \) marks

Let \( V \) be a vector space over a field \( F \), and let \( S = \{v_1, \ldots, v_m\} \) be a finite subset of \( V \).

(i) What does it mean to say that \( S \) is linearly independent? \( 1 \) mark

(ii) What is meant by \( \text{Span}(S) \), the span of \( S \)? \( 1 \) mark

(iii) Consider the subset \( S := \{\cos^2 x, \sin^2 x, \sin 2x\} \) of the \( \mathbb{R} \)-vector space \( C(\mathbb{R}) \) of continuous real-valued functions of a real variable. Prove that the constant function 1 belongs to \( \text{Span}(S) \). Prove also that \( S \) is linearly independent. (Hint: in a putative linear dependence relation, plug in carefully chosen values of \( x \) to show that all the coefficients have to be 0.) Is \( \{\cos^2 x, \sin^2 x, \cos 2x\} \) linearly independent? \( 4 \) marks

By solving a set of linear equations, find a basis for the kernel of the linear map \( \ell : \mathbb{R}^3 \rightarrow \mathbb{R}^2 \) defined by \( \ell(x) := Ax \), where \( A := \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \end{pmatrix} \). \( 2 \) marks
Suppose that for some field $F$, and some integer $n \geq 1$, $A, B \in M_n(F)$, with $B$ invertible.

(i) Suppose that $v \in F^n$ is an eigenvector for $A$, with eigenvalue $\lambda$. Prove that $Bv$ is an eigenvector for $BAB^{-1}$, with eigenvalue $\lambda$. (Just multiply the matrix by the vector and see what happens.) \hfill (1 mark)

(ii) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Let $f_A, f_B \in L(\mathbb{R}^3)$ be the linear operators $x \mapsto Ax$, $x \mapsto Bx$, respectively.

(a) Describe each of $f_A$ and $f_B$ as a rotation through a certain angle about a certain axis. \hfill (2 marks)

(b) Starting from the characteristic polynomial $\det(xI - A)$, show that if $v \in \mathbb{R}^3$ is a (real) eigenvector for $f_A$, then $v$ is unique, up to scalar multiples. Using (i), find an eigenvector for $BAB^{-1}$. How can we describe $f_{BAB^{-1}}$ geometrically? \hfill (4 marks)

On the space $C^\infty(\mathbb{R}_{>0}, \mathbb{R})$ of real-valued functions on the domain $\mathbb{R}_{>0}$, with derivatives of all orders, consider the linear operator $\ell$ defined by $\ell(y)(x) := x^2 \frac{dy}{dx}$. By solving a differential equation, find an eigenvector for $\ell$, with eigenvalue 2. \hfill (2 marks)

Let $V$ be a vector space over $\mathbb{R}$, with an inner product $\langle \cdot, \cdot \rangle$. Let $T \in L(V)$ be a linear operator (i.e. a linear map from $V$ to $V$).

(i) What does it mean for $T$ to be self-adjoint with respect to $\langle \cdot, \cdot \rangle$? \hfill (1 mark)

(ii) Suppose that $T$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$, and that $v_1, v_2$ are eigenvectors for $T$, with eigenvalues $\lambda_1, \lambda_2$ respectively. (So $Tv_1 = \lambda_1 v_1$ and $Tv_2 = \lambda_2 v_2$, with $v_1, v_2$ non-zero.) Prove that if $\lambda_1 \neq \lambda_2$ then $v_1$ and $v_2$ are orthogonal with respect to $\langle \cdot, \cdot \rangle$. \hfill (2 marks)

(iii) Now let $V = \mathbb{R}^n$, with the standard dot product, i.e.

$$\langle x, y \rangle = x \cdot y := x^t y = \sum_{i=1}^{n} x_i y_i, \text{ for any } x, y \in \mathbb{R}^n.$$ 

Let $T \in L(\mathbb{R}^n)$ be defined by $T(x) = Ax$, where $A \in M_n(\mathbb{R})$ is a symmetric matrix. Prove that $T$ is self-adjoint with respect to $\langle \cdot, \cdot \rangle$. \hfill (2 marks)

(iv) Use the special case $n = 2$, $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, finding the eigenvalues and eigenvectors, to illustrate what you have proved in (ii). \hfill (3 marks)
12 You may assume that $\langle f, g \rangle := \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1 - x^2}} \, dx$ defines an inner product on the $\mathbb{R}$-vector space $\mathbb{R}[x]$ of polynomials in one variable with real coefficients.

(i) Show that 1 and $x$ are orthogonal with respect to this inner product. Find the norms of 1, $x$ and $3+2x$. (For the norm of $x$, you might like to consider a trigonometric substitution.) \hspace{1cm} \text{(5 marks)}

(ii) Explain why $\langle f, g \rangle := \int_{-1}^{1} x f(x)g(x) \, dx$ does not define an inner product on $\mathbb{R}[x]$. \hspace{1cm} \text{(2 marks)}

End of Question Paper