A (delayed) renewal process is defined by tossing a biased coin with probability $p$ (where $0 < p < 1$) of giving a head repeatedly, and saying that a renewal occurs whenever a run of two consecutive heads is completed. You should assume that the results of different tosses are independent.

(a) Let $u_n$ be the probability that, given that a renewal occurs at time $t$, a renewal occurs at time $t + n$. Explain why $u_1 = p$, and give the value of $u_n$ for $n \geq 2$. Hence show that the generating function $U(s)$, defined as

$$U(s) = \sum_{n=0}^{\infty} u_n s^n$$

for $|s| < 1$, has the form

$$U(s) = 1 + ps + \frac{p^2 s^2}{1 - s}.$$  

(7 marks)

(b) Let $v_n$ be the probability that a renewal occurs at time $n$. Give the values of $v_0$ and $v_1$, and explain why $v_n = p^2$ for $n \geq 2$. Hence show that the generating function $V(s)$, defined as

$$V(s) = \sum_{n=0}^{\infty} v_n s^n$$

for $|s| < 1$, has the form

$$V(s) = \frac{p^2 s^2}{1 - s}.$$  

(6 marks)

(c) Let $f_n$ be the probability that, given that a renewal occurs at time $t$, the next renewal occurs at time $t + n$. What is the value of $f_2$? Explain your answer carefully.  

(3 marks)

(d) Using the result that, in a delayed renewal process, $V(s) = U(s)B(s)$, where $B(s)$ is the probability generating function of the time until the first renewal, find the expected number of tosses until the first renewal.  

(5 marks)
Let \((Y_n)\) be a Markov chain on \(S = \{1, 2, 3, 4\}\) with transition matrix

\[
P = \begin{pmatrix}
\frac{1}{3} & 1 & \frac{1}{3} & 0 \\
\frac{1}{3} & 0 & \frac{1}{3} & 1 \\
0 & \frac{1}{3} & 0 & \frac{1}{3} \\
0 & 0 & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}.
\]

(a) Find the communicating classes of the Markov chain, and state which states are transient and which are recurrent. \((6 \text{ marks})\)

(b) Find all stationary distributions of the Markov chain. \((7 \text{ marks})\)

(c) Assume the chain starts at time 0 in state 1.

(i) Prove by induction that, for \(n \geq 0\), \(P(Y_n = 1) = \left(\frac{1}{3}\right)^n\) and \(P(Y_n = 2) = \frac{1}{2} \left(1 - \left(\frac{1}{3}\right)^n\right)\). \((6 \text{ marks})\)

(ii) Assuming that the distribution of \(Y_n\) converges to a stationary distribution of the chain, which stationary distribution will that be? \((2 \text{ marks})\)

Let \((X_n)\) be a Markov chain on \(S = \{1, 2, 3, 4, 5\}\) with transition matrix

\[
P = \begin{pmatrix}
0 & p & 0 & 0 & 1 - p \\
1 - p & 0 & p & 0 & 0 \\
0 & 1 - p & 0 & p & 0 \\
0 & 0 & 1 - p & 0 & p \\
p & 0 & 0 & 1 - p & 0
\end{pmatrix}.
\]

(a) Assume \(0 < p < 1\).

(i) Show that the chain is irreducible and aperiodic. \((6 \text{ marks})\)

(ii) Verify that \(\left(\frac{1}{5} \frac{1}{5} \frac{1}{5} \frac{1}{5} \frac{1}{5}\right)\) is a stationary distribution for the chain. \((2 \text{ marks})\)

(iii) Hence prove that, for all \(i \in S\), \(P(X_n = i) \to \frac{1}{5}\) as \(n \to \infty\). You may use results from the course. \((6 \text{ marks})\)

(b) Assume \(p = 1\).

(i) Is the chain still irreducible? Explain your answer carefully. \((3 \text{ marks})\)

(ii) Is the chain still aperiodic? Explain your answer carefully. \((3 \text{ marks})\)
A gambler repeatedly plays a game where the player wins £2 with probability 1/3 and loses £1 with probability 2/3. The gambler has an initial fortune of £1, and will stop playing if and only if either he runs out of money or his fortune reaches at least £4, in which case you should think of his fortune as remaining the same in the future. Modelling the gambler’s fortune as a Markov chain with state space \{0, 1, 2, 3, 4, 5\}:

(a) Give the transition matrix of the Markov chain. \(4 \text{ marks}\)

(b) Find the probability that the gambler runs out of money. \(7 \text{ marks}\)

(c) Find the expected number of games the gambler plays before stopping playing. \(7 \text{ marks}\)

Assume that emails arrive in an account as a Poisson process with rate 4 per hour.

(a) What is the distribution of the number of emails which arrive between 9am and 11am on a given day? \(3 \text{ marks}\)

(b) What is the probability that no emails arrive between 9am and 10am on a given day? \(2 \text{ marks}\)

(c) Given that six emails arrive between 9am and 11am, what is the probability that exactly one of them arrives before 10am? \(4 \text{ marks}\)

(d) Show that the distribution function of \(S_2\), the amount of time (in hours) after 9am that the second email arrives, is \(1 - e^{-4t}(1 + 4t)\) for \(t > 0\). \(6 \text{ marks}\)

(e) Each email is marked as spam with probability 3/4, independently of other emails. Describe the process of arrivals of non-spam emails, and give a reason for your answer. \(5 \text{ marks}\)

End of Question Paper