The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2015–2016**

Probability Modelling **2 hours**

*Candidates should attempt **ALL** five questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–21; Q2–21; Q3–20; Q4–18; Q5–20)

1 A (delayed) renewal process is defined by tossing a biased coin with probability p (where $0 < p < 1$) of giving a head repeatedly, and saying that a renewal occurs whenever a run of two consecutive heads is completed. You should assume that the results of different tosses are independent.

(a) Let u_n be the probability that, given that a renewal occurs at time t, a renewal occurs at time $t+n$. Explain why $u_1 = p$, and give the value of u_n for $n \geq 2$. Hence show that the generating function $U(s)$, defined as $\sum_{n=0}^{\infty} u_n s^n$ for $|s| < 1$, has the form

$$U(s) = 1 + ps + \frac{p^2 s^2}{1-s}.$$

(7 marks)

(b) Let v_n be the probability that a renewal occurs at time n. Give the values of v_0 and v_1, and explain why $v_n = p^2$ for $n \geq 2$. Hence show that the generating function $V(s)$, defined as $\sum_{n=0}^{\infty} v_n s^n$ for $|s| < 1$, has the form

$$V(s) = \frac{p^2 s^2}{1-s}.$$

(6 marks)

(c) Let f_n be the probability that, given that a renewal occurs at time t, the next renewal occurs at time $t+n$. What is the value of f_2? Explain your answer carefully. ***(3 marks)***

(d) Using the result that, in a delayed renewal process, $V(s) = U(s)B(s)$, where $B(s)$ is the probability generating function of the time until the first renewal, find the expected number of tosses until the first renewal.

(5 marks)

2 Let (Y_n) be a Markov chain on $S = \{1, 2, 3, 4\}$ with transition matrix

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

(a) Find the communicating classes of the Markov chain, and state which states are transient and which are recurrent. ***(6 marks)***

(b) Find all stationary distributions of the Markov chain. ***(7 marks)***

(c) Assume the chain starts at time 0 in state 1.

(i) Prove by induction that, for $n \geq 0$, $P(Y_n = 1) = \left(\frac{1}{3}\right)^n$ and $P(Y_n = 2) = \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$. ***(6 marks)***

(ii) Assuming that the distribution of Y_n converges to a stationary distribution of the chain, which stationary distribution will that be? ***(2 marks)***

3 Let (X_n) be a Markov chain on $S = \{1, 2, 3, 4, 5\}$ with transition matrix

$$P = \begin{pmatrix} 0 & p & 0 & 0 & 1-p \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ p & 0 & 0 & 1-p & 0 \end{pmatrix}.$$

(a) Assume $0 < p < 1$.

(i) Show that the chain is irreducible and aperiodic. ***(6 marks)***

(ii) Verify that $\begin{pmatrix} \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$ is a stationary distribution for the chain. ***(2 marks)***

(iii) Hence prove that, for all $i \in S$, $P(X_n = i) \to \frac{1}{5}$ as $n \to \infty$. You may use results from the course. ***(6 marks)***

(b) Assume $p = 1$.

(i) Is the chain still irreducible? Explain your answer carefully. ***(3 marks)***

(ii) Is the chain still aperiodic? Explain your answer carefully. ***(3 marks)***

4 A gambler repeatedly plays a game where the player wins £2 with probability 1/3 and loses £1 with probability 2/3. The gambler has an initial fortune of £1, and will stop playing if and only if either he runs out of money or his fortune reaches at least £4, in which case you should think of his fortune as remaining the same in the future. Modelling the gambler's fortune as a Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$:

(a) Give the transition matrix of the Markov chain. ***(4 marks)***

(b) Find the probability that the gambler runs out of money. ***(7 marks)***

(c) Find the expected number of games the gambler plays before stopping playing. ***(7 marks)***

5 Assume that emails arrive in an account as a Poisson process with rate 4 per hour.

(a) What is the distribution of the number of emails which arrive between 9am and 11am on a given day? ***(3 marks)***

(b) What is the probability that no emails arrive between 9am and 10am on a given day? ***(2 marks)***

(c) Given that six emails arrive between 9am and 11am, what is the probability that exactly one of them arrives before 10am? ***(4 marks)***

(d) Show that the distribution function of S_2, the amount of time (in hours) after 9am that the second email arrives, is $1 - e^{-4t}(1 + 4t)$ for $t > 0$. ***(6 marks)***

(e) Each email is marked as spam with probability 3/4, independently of other emails. Describe the process of arrivals of non-spam emails, and give a reason for your answer. ***(5 marks)***

End of Question Paper