Data provided: Formula sheet

MAS323

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2015–2016

Differential Equations: A Case Study — **2 hours**

Attempt all questions.

1 Consider the system of equations

$$\frac{dx}{dt} = -y + xF(r), \qquad \frac{dy}{dt} = x + yF(r). \qquad (1)$$

where $r^2 = x^2 + y^2$, and $F(r)$ is a real-valued continuous function of r.

(i) Show that $x = 0$, $y = 0$ is the only critical point of the system. By considering the linearised system, show that this critical point is a spiral. What is the nature of the critical point if $F(0) > 0$? ***(8 marks)***

(ii) Use the variable substitution

$$x = r\cos\theta, \qquad y = r\sin\theta,$$

to show that the system (1) can be written as

$$\frac{dr}{dt} = rF(r), \qquad \frac{d\theta}{dt} = 1.$$

Thus show that system (1) has a periodic solution for each value r_0 of r such that $F(r_0) = 0$. ***(5 marks)***

(iii) By considering a small perturbation, $r = r_0 + \delta$, about r_0 and using a Taylor expansion of function $F(r)$, show that this periodic solution is a stable limit cycle in the case $F'(r_0) < 0$, and it is an unstable limit cycle in the case $F'(r_0) > 0$. ***(5 marks)***

(iv) Find all limit cycles of the system

$$\frac{dx}{dt} = x - y - x\sqrt{x^2 + y^2}, \qquad \frac{dy}{dt} = x + y - y\sqrt{x^2 + y^2},$$

and determine their stability. Sketch a phase portrait for the system. ***(7 marks)***

2 A model of two interacting species, x and y, is described by the system of equations

$$\begin{aligned}\frac{dx}{dt} &= N_0\,x\left(1-\frac{x}{K_0}-\frac{y}{K_1}\right),\\ \frac{dy}{dt} &= N_1\,y\left(1-\frac{x}{K_2}-\frac{y}{K_3}\right),\end{aligned}$$

where all parameters are positive, $x \geq 0$ and $y \geq 0$.

(i) What kind of interaction does the model describe? What are the meanings of the parameters K_0 and K_3? ***(2 marks)***

(ii) Use the variable substitutions

$$X=\frac{x}{K_0},\quad Y=\frac{y}{K_3},\quad T=N_0 t,$$

to show that the system can be written in the dimensionless form

$$\begin{aligned}\frac{dX}{dT} &= X(1-X-\beta_0 Y) \equiv f(X,Y),\\ \frac{dY}{dT} &= \rho Y(1-Y-\beta_1 X) \equiv g(X,Y),\end{aligned}$$

and find expressions for ρ, β_0 and β_1 in terms of the model parameters. ***(5 marks)***

(iii) Find all the critical points for the system, stating carefully any conditions which must be met for their existence. Determine how the stability of the critical points that correspond to one of the species forcing the other to extinction depends on β_0 and β_1. ***(9 marks)***

(iv) Sketch the nullclines of the system when (a) $\beta_0 > 1$ and $\beta_1 > 1$, and (b) $\beta_0 < 1$ and $\beta_1 < 1$. In each case, add examples of trajectories to your sketch. Explain how the system will behave in each case. For what values of ρ, β_0 and β_1 can the two species coexist at a stable steady state? ***(9 marks)***

3 Consider a spatial model for a harvested fish population

$$\begin{aligned}\frac{\partial U}{\partial t} &= rU\left(1-\frac{U}{K}\right) - EU + D\frac{\partial^2 U}{\partial x^2}, \quad x \geq 0,\\ \frac{\partial U}{\partial x} &= 0 \quad \text{when } x = 0,\end{aligned}$$

where $U(x,t)$ is the fish population level at a distance x from a shoreline, and D, r, K and E are positive constants.

(i) Describe briefly the biological meaning of the terms in the model, and the meaning of the boundary condition at $x = 0$. ***(4 marks)***

(ii) Show that the model has a non-zero *spatially uniform* steady state $U_* > 0$ only if $E < r$. Assuming $E < r$, find U_* and determine whether or not the state $U(x,t) = U_*$ is stable to spatially uniform perturbations. ***(6 marks)***

(iii) Now assume that fishing is regulated such that $E = E_0 < r$ in the region $0 \leq x < H$. Also assume that $E \gg r$ in the region $x \geq H$, such that we can approximate $U(x,t) = 0$ for $x \geq H$. Linearise the system about the steady state $U = 0$. Considering small perturbations of the form

$$U(x,t) = e^{at}\left(\alpha\cos bx + \beta\sin bx\right), \tag{2}$$

show that the boundary conditions at $x = 0$ and $x = H$ imply that $\beta = 0$ and

$$b = \frac{(2n+1)\pi}{2H}, \quad n = 0, 1, 2, \ldots$$

(7 marks)

(iv) Show that spatial perturbations of the form (2) can grow only if

$$H > \frac{\pi}{2}\sqrt{\frac{D}{r - E_0}}$$

and sketch the spatial form of the fastest growing perturbation to the steady state. ***(8 marks)***

4 (i) The Euler-Lagrange equation corresponding to a functional $f(x, y(x), y'(x))$ is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(f - y'\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial x}.$$

If f is independent of x, show that

$$f - y'\frac{\partial f}{\partial y'} = \text{constant}.$$

(6 marks)

(ii) Consider two points A and B with coordinates $(-1, 1)$ and $(1, 1)$ respectively. Show that the functional

$$I = \int_A^B \frac{\sqrt{1 + y'^2}}{y}\,\mathrm{d}x$$

achieves an extremal value when $x^2 + y^2 = 2$. Sketch this path between A and B. ***(10 marks)***

(iii) Show that the value of I evaluated along this path is $I_0 = 2\ln(1 + \sqrt{2})$. ***(6 marks)***

(iv) By evaluating I along the straight line path between A and B, show that I_0 is a minimal value for I. ***(3 marks)***

End of Question Paper

List of Basic Formulae and Theorems

Theorem 1: If a periodic solution of the system of equations

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y)$$

exists in a simply connected region, then $f_x + g_y = 0$ somewhere in that region.

Corollary: There are no periodic solutions in any simply connected region where $f_x + g_y \neq 0$ everywhere.

Theorem 2: The orbit $\mathcal{C}$ of a periodic solution must enclose at least one critical point.

Orthogonality conditions for trig functions

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = 0, \quad \int_{-\pi}^{\pi} \cos mx \cos nx \, dx = 0 \quad \text{when } \ m \neq n.$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0.$$

Extremals of functional

$$J[y] = \int_{x_0}^{x_1} f(y, y', x) \, dx$$

are the solutions to the Euler-Lagrange equation

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$