The Fourier transform, \( \hat{f}(k) \), of a function \( f(x) \) is defined by

\[
\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x) \, dx.
\]

(a) Write down the inverse Fourier transform, and use it to derive Parseval’s theorem:

\[
\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{f}(k)|^2 \, dk. \tag{9 marks}
\]

(b) The function \( f(x) \) is defined by

\[
f(x) = \begin{cases} 
  x & |x| \leq 1 \\
  0 & |x| > 1.
\end{cases}
\]

Find \( \hat{f}(k) \), and use Parseval’s theorem to deduce that

\[
\int_{0}^{\infty} \left( \sin k - k \cos k \right)^2 \, dk = \frac{\pi}{6}. \tag{16 marks}
\]
The Laplace transform of a function \( f(t) \) is defined by
\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt.
\]

(a) Find the Laplace transform of \( \sin \omega t \) for \( \Re s > 0 \), where \( \omega \) is a real constant. \( \text{(5 marks)} \)

(b) The function \( f(t) \) is defined by
\[
f(t) = \begin{cases} 
1 - t & 0 \leq t \leq 1 \\
0 & t > 1.
\end{cases}
\]
Find the Laplace transform of \( f(t) \). \( \text{(5 marks)} \)

(c) \( F(t) \) is defined for \( t > 0 \) by
\[
F(t) = \int_0^t f(\tau) g(t - \tau) \, d\tau.
\]
Find \( F(t) \) for \( t > 0 \) if \( f(t) \) is as defined in part (b), and \( g(t) = \sin \omega t \) where \( \omega \) is a real constant. \( \text{(11 marks)} \)

(d) Given that for \( t > 0 \)
\[
\mathcal{L}\{F(t)\} = \mathcal{L}\{f(t)\} \mathcal{L}\{g(t)\},
\]
use the results of parts (a), (b) and (c) to deduce that if \( \omega \) is a real constant then the inverse Laplace transform of
\[
\frac{\omega(s + e^{-s} - 1)}{s^2(s^2 + \omega^2)}
\]
is
\[
-\frac{1}{\omega} \cos \omega t + \frac{1}{\omega^2} \sin \omega t - \frac{H(t - 1)}{\omega^2} \sin \omega(t - 1) + \frac{H(1 - t)}{\omega} (1 - t) \quad \text{for } t > 0,
\]
where \( H \) is the Heaviside step function. \( \text{(4 marks)} \)
3 The function $y(x)$ satisfies the ordinary differential equation

$$x^2 y'' - 2y = \ln(1 + x) \quad 0 \leq x \leq 1,$$

with the boundary conditions

$y$ finite as $x \to 0$

$y = 0$ at $x = 1$.

(a) By trying $y = x^n$, find the independent solutions of

$$x^2 y'' - 2y = 0.$$  

(3 marks)

(b) Given that Green’s function $G(x; \xi)$ for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x; \xi) = \begin{cases} 
\frac{1}{3} \left( 1 - \frac{1}{\xi^3} \right) x^2 & 0 \leq x < \xi, \\
\frac{1}{3} \left( x^2 - \frac{1}{x} \right) & \xi < x \leq 1.
\end{cases}$$  

(14 marks)

(c) Use Green’s function to write down the solution to equation (1) and the boundary conditions given at the beginning of the question (do NOT attempt the $\xi$ integrals).

Use this to show that

$$y'(1) = 2 \ln 2 - 1.$$  

(5 marks)

4 Consider the equation

$$(1 - \epsilon)x^2 + 4x + 4 = 0,$$

where $\epsilon$ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation (2) can be written as

$$x = x_0 + \epsilon^{1/2} x_1 + \epsilon x_2 + \epsilon^{3/2} x_3 + \epsilon^2 x_4 + \epsilon^{5/2} x_5 + \cdots,$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to derive the two solutions to equation (2), correct to order $\epsilon^2$ as $\epsilon \to 0$.  

(20 marks)

(b) Find the exact solutions of (2), and show that their expansions agree with your results from part (a).  

(5 marks)
The Laplace transform, \( \hat{f}(s) \), of a function \( f(t) \) is defined by

\[
\hat{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt,
\]

where \( s \) is real and positive.

(a) By integrating by parts, show that if \( f \) is \( n \) times differentiable then

\[
\hat{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \cdots + \frac{f^{(n-1)}(0)}{s^n} + R_n(s),
\]

where

\[
R_n(s) = \frac{1}{s^n} \int_0^\infty e^{-st} f^{(n)}(t) \, dt.
\]

You may assume that \( \lim_{t \to \infty} \{e^{-st} f^{(m)}(t)\} = 0 \) for \( m = 0, 1, \ldots, n - 1 \).

(b) Use equation (3) to expand \( \hat{f}(s) \) for \( f(t) = \ln(1 + 2t) \).

Show that in this case

\[
|R_n(s)| < \frac{2^n(n - 1)!}{s^{n+1}},
\]

and, hence, that the expansion is an asymptotic expansion as \( s \to \infty \).

End of Question Paper