SCHOOL OF MATHEMATICS AND STATISTICS  
Mathematics (Computational Methods)  

Marks will be awarded for your best FOUR answers
1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic.

(a) \[ \frac{\partial u}{\partial p} + \frac{\partial^2 u}{\partial x \partial p} - 2 \frac{\partial^2 u}{\partial p^2} = -\frac{\partial u}{\partial x^2} \] (1 mark)

(b) \[ \frac{\partial^2 u}{\partial x^2} + 2 \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial u^2}{\partial y^2} - 3u \] (1 mark)

(c) \[ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x} = 4 \] (1 mark)

(ii) Consider the following Advection-Diffusion equation

\[ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 2 \frac{\partial^2 u}{\partial x^2}. \]

Write down the general recurrence relation for the *implicit* scheme. (5 marks)

(iii) Consider the following version of the heat-diffusion equation.

\[ \frac{\partial u}{\partial t} = \frac{1}{8} \frac{\partial^2 u}{\partial x^2}. \]

Consider the following boundary conditions

\[ u(x, 0) = 1 - x^2, \quad u(1, t) = 0, \quad \frac{\partial u(0, t)}{\partial x} = -1. \]

Using the *explicit* difference scheme, solve the differential equation between \(0 \leq x \leq 1, 0 \leq t \leq 0.75\) for

\[ \Delta x = h = \frac{1}{3}, \quad \Delta t = k = \frac{1}{4}. \] (15 marks)

(iv) Comment on the physicality of your solution to part (iii). (2 marks)
2 (i) Formulate the following sets of constraints in a form suitable for integer programming:
(a) 
\[4x_1 + 2x_2 \leq 9, \quad 5x_3 - x_2 \leq 10, \quad x_1 + 4x_2 \leq 12,\]
where 2 out of 3 constraints hold. \((3 \text{ marks})\)
(b) 
Either \(2x_1 + 5x_2 \leq 13,\) or \(3x_1 + 2x_2 \leq 12\) holds. \((2 \text{ marks})\)

(ii) Use the Branch & Bound algorithm (with tree diagrams) to solve the following IP problems;
\[z = 3x_1 + 4x_2,\]
subject to
\[2x_1 + 3x_2 \leq 18, \quad 2x_1 + x_2 \leq 12 \quad x_1, x_2 \geq 0.\]
\((15 \text{ marks})\)

(iii) Graphically represent the feasible areas. \((5 \text{ marks})\)

3 Consider the following expressions
\[f(x, y) = \frac{2}{3}x^3 + 2x^2y - y^2 + 2xy^2, \quad (1)\]
and
\[g(x, y) = x^4 + 4xy + y^4 + 6x^3y + 6xy^3. \quad (2)\]
(i) Find and classify all stationary points of \(f(x, y)\) in (1). \((6 \text{ marks})\)

(ii) Discuss the advantages and disadvantages of the steepest descent and Newtons methods in finding the roots of \(g(x, y)\) in (2). \((3 \text{ marks})\)

(iii) State a reason to suggest why \(g(x, y)\) in (2) has a minimum. \((1 \text{ mark})\)

(iv) Apply one iteration of the method of steepest descent to \(g\), from (2), starting from the point \((1,1)\). \((6 \text{ marks})\)

(v) Comment on your answer to part (iv). \((2 \text{ marks})\)

(vi) Determine the Newton step for \(g\), as given in (2), from the starting point of \((1,1)\). \((7 \text{ marks})\)
4 (i) Contrast the method of cubic splines with the Lagrange and linear interpolation methods. (3 marks)

(ii) Write down the values for $\sigma_0$ and $\sigma_1$ under the assumption of

$$f'(x_0) = 0, \quad f''(x_1) = 0.$$  

(2 marks)

(iii) Using the conditions derived in (ii), determine the cubic spline between the following data points

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\pi$</th>
<th>$3\pi/2$</th>
<th>$2\pi$</th>
<th>$5\pi/2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>1</td>
<td>0.5</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(18 marks)

(iv) Determine the value of the cubic spline at $f(5\pi/3)$. (2 marks)

5 A manufacturer uses a certain machine and wishes to estimate operational costs for a 9 year period. He decides that, at the end of each year, the machine will either be overhauled or replaced. New machines cost £2000 whilst the cost of an overhaul is related to machine age (at year end) as follows:

<table>
<thead>
<tr>
<th>Age(Years)</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost(£100)</td>
<td>3</td>
<td>8</td>
<td>15</td>
</tr>
</tbody>
</table>

Machines which are four years old are regarded as scrap and are replaced automatically. If the manufacturer begins the first year with a new machine, use the DP method to find the minimal operational cost over the nine year period - excluding the initial cost of the new machine. Given that the manufacturer always keeps a machine when the optimal plan gives him a choice and neglecting the choice in the final year, determine the years for which a new machine must be bought. (25 marks)

End of Question Paper
Formulae Sheet

Notation:

\[ U(x_i, t_j) \equiv U_{ij} \]

Forward difference formula for \( \frac{\partial U}{\partial t} \):

\[ \frac{\partial U}{\partial t} \approx \frac{U_{ij+1} - U_{ij}}{\Delta t} \]

Backward difference formula for \( \frac{\partial U}{\partial t} \):

\[ \frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{ij-1}}{\Delta t} \]

Central difference formula for \( \frac{\partial U}{\partial x} \):

\[ \frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x} \]

Central difference formula for \( \frac{\partial^2 U}{\partial x^2} \):

\[ \frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2} \]