Data provided: Formulae sheet

MAS340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2015-2016

Mathematics (Computational Methods) — Two hours

Marks will be awarded for your best FOUR answers

1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic.

(a) $\frac{\partial u}{\partial p} + \frac{\partial^2 u}{\partial x \partial p} - 2\frac{\partial^2 u}{\partial p^2} = -\frac{\partial^2 u}{\partial x^2}$ ***(1 mark)***

(b) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial u^2}{\partial y^2} - 3u$ ***(1 mark)***

(c) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y \partial x} = 4$ ***(1 mark)***

(ii) Consider the following Advection-Diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 2\frac{\partial^2 u}{\partial x^2}.$$

Write down the general recurrence relation for the *implicit* scheme.

(5 marks)

(iii) Consider the following version of the heat-diffusion equation.

$$\frac{\partial u}{\partial t} = \frac{1}{8}\frac{\partial^2 u}{\partial x^2}$$

Consider the following boundary conditions

$$u(x,0) = 1 - x^2, \qquad u(1,t) = 0, \qquad \frac{\partial u(0,t)}{\partial x} = -1.$$

Using the *explicit* difference scheme, solve the differential equation between $0 \leq x \leq 1, 0 \leq t \leq 0.75$ for

$$\Delta x = h = \frac{1}{3}, \qquad \Delta t = k = \frac{1}{4}.$$

(15 marks)

(iv) Comment on the physicality of your solution to part (iii). ***(2 marks)***

2 (i) Formulate the following sets of constraints in a form suitable for integer programming:

(a)

$$4x_1 + 2x_2 \leq 9, \qquad 5x_1 - x_2 \leq 10, \qquad x_1 + 4x_2 \leq 12,$$

where 2 out of 3 constraints hold. ***(3 marks)***

(b)

$$\text{Either} \qquad 2x_1 + 5x_2 \leq 13, \qquad \text{or} \qquad 3x_1 + 2x_2 \leq 12 \qquad \text{holds.}$$

(2 marks)

(ii) Use the Branch & Bound algorithm (with tree diagrams) to solve the following IP problems;

$$z = 3x_1 + 4x_2,$$

subject to

$$2x_1 + 3x_2 \leq 18, \qquad 2x_1 + x_2 \leq 12 \qquad x_1, x_2 \geq 0.$$

(15 marks)

(iii) Graphically represent the feasible areas. ***(5 marks)***

3 Consider the following expressions

$$f(x, y) = \frac{2}{3}x^3 + 2x^2y - y^2 + 2xy^2, \tag{1}$$

and

$$g(x, y) = x^4 + 4xy + y^4 + 6x^3y + 6xy^3. \tag{2}$$

(i) Find and classify all stationary points of $f(x, y)$ in (1). ***(6 marks)***

(ii) Discuss the advantages and disadvantages of the steepest descent and Newtons methods in finding the roots of $g(x, y)$ in (2). ***(3 marks)***

(iii) State a reason to suggest why $g(x, y)$ in (2) has a minimum. ***(1 mark)***

(iv) Apply one iteration of the method of steepest descent to g, from (2), starting from the point (1,1). ***(6 marks)***

(v) Comment on your answer to part (iv). ***(2 marks)***

(vi) Determine the Newton step for g, as given in (2), from the starting point of (1,1). ***(7 marks)***

4 (i) Contrast the method of cubic splines with the Lagrange and linear interpolation methods. ***(3 marks)***

(ii) Write down the values for σ_0 and σ_n under the assumption of

$$f'(x_0) = 0, \qquad f''(x_n) = 0.$$

(2 marks)

(iii) Using the conditions derived in (ii), determine the cubic spline between the following data points

x	π	$3\pi/2$	2π	$5\pi/2$
f(x)	1	0.5	0	1

(18 marks)

(iv) Determine the value of the cubic spline at f($5\pi/3$). ***(2 marks)***

5 A manufacturer uses a certain machine and wishes to estimate operational costs for a 9 year period. He decides that, at the end of each year, the machine will either be overhauled or replaced. New machines cost £2000 whilst the cost of an overhaul is related to machine age (at year end) as follows:

Age(Years)	1	2	3
Cost(£100)	3	8	15

Machines which are four years old are regarded as scrap and are replaced automatically. If the manufacturer begins the first year with a new machine, use the DP method to find the minimal operational cost over the nine year period - excluding the initial cost of the new machine. Given that the manufacturer always keeps a machine when the optimal plan gives him a choice and neglecting the choice in the final year, determine the years for which a new machine must be bought.
(25 marks)

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$