SCHOOL OF MATHEMATICS AND STATISTICS

Applicable Analysis

Spring Semester 2015–2016

Answer four questions. If you answer more than four questions, only your best four will be counted.

You may use the following results when answering questions on this paper.

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t^\alpha e^{bt}$</td>
<td>$\frac{\Gamma(\alpha + 1)}{(s - b)^{\alpha + 1}}$</td>
</tr>
<tr>
<td>$\sin at$</td>
<td>$\frac{a}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$\cos at$</td>
<td>$\frac{s}{s^2 + a^2}$</td>
</tr>
<tr>
<td>$f(t)e^{bt}$</td>
<td>$F(s - b)$</td>
</tr>
<tr>
<td>$f^{(n)}(t)$</td>
<td>$s^n F(s) - \sum_{k=1}^{n} f^{(k-1)}(0)s^{n-k}$</td>
</tr>
<tr>
<td>$tf(t)$</td>
<td>$-F'(s)$</td>
</tr>
</tbody>
</table>

Please leave this exam paper on your desk

Do not remove it from the hall

Registration number from U-Card (9 digits) to be completed by student
1 (i) Define what is meant by the statement that the improper integral
\[ \int_a^\infty f(x) \, dx \] exists. \hspace{1cm} (2 marks)

(a) Prove from your definition that \( \int_0^\infty 4 \sin x \cos x \, dx \) does not exist. \hspace{1cm} (2 marks)

(b) Using your definition, find the values of \( \alpha \) for which
\[ \int_0^\infty \frac{2x}{(1 + x^2)^\alpha} \, dx \] exists. \hspace{1cm} (5 marks)

(ii) State, without proof, the Comparison Test for convergence and divergence of integrals of the form \( \int_a^\infty f(x) \, dx \). Your statement should include conditions under which the results are valid. \hspace{1cm} (4 marks)

Determine whether each of the following integrals converges or diverges, giving reasons for your answers.

(c) \[ \int_0^\infty \frac{2x \cos x}{(x^2 + 1)^4} \, dx , \]

(d) \[ \int_1^\infty \frac{1}{(x^3 + 7)^{\frac{1}{3}}} \, dx . \]

(7 marks)

(iii) (e) Determine whether
\[ \int_0^1 \frac{e^x \cos x}{(1 + x^2)^{\sqrt{x}}} \, dx \] converges or diverges, giving reasons for your answer.

(f) Determine whether
\[ \int_0^1 \frac{e^x \cos x}{x \sqrt{x}} \, dx \] converges or diverges, giving reasons for your answer. \hspace{1cm} (5 marks)
2 (i) State, without proof, the theorem concerning change of order in a repeated
integral of the form
\[ \int_c^d dy \int_a^\infty f(x, y) \, dx. \]
Your statement should include conditions under which the result holds.  \( (2 \text{ marks}) \)

Let \( 0 < c < d \). Prove that
\[ \int_c^d dy \int_0^\infty \frac{2y}{4x^2 + y^2} \, dx = \int_0^\infty dx \int_c^d \frac{2y}{4x^2 + y^2} \, dy. \]  \( (6 \text{ marks}) \)

Deduce that
\[ \int_0^\infty \ln \left( \frac{4x^2 + d^2}{4x^2 + c^2} \right) \, dx = \frac{\pi(d-c)}{2}. \]  \( (7 \text{ marks}) \)

(ii) Define the \( \Gamma \) function.  \( (2 \text{ marks}) \)

Prove that
\[ (a) \int_1^\infty \frac{(\ln x)^5}{x\sqrt{x}} \, dx = 2^6(5!) \]  \( (4 \text{ marks}) \)
and
\[ (b) \int_0^\infty x^9 e^{-4x^4} \, dx = \frac{3\sqrt{\pi}}{2^9}. \]  \( (4 \text{ marks}) \)
Define the Beta function. State, without proof, the relation between the Beta and Gamma functions. (3 marks)

Prove that

\[ B(x, y) = 2 \int_0^{\pi/2} \cos^{2x-1} \theta \sin^{2y-1} \theta \, d\theta \quad (x > 0, y > 0) \]

and

\[ B(x, y) = \int_0^\infty \frac{u^{x-1}}{(1 + u)^{x+y}} \, du \quad (x > 0, y > 0). \] (4 marks)

Prove each of the following, stating any standard results you need to use:

(a) \[ \int_0^{\pi/2} \frac{1}{\sqrt{\tan \theta}} \, d\theta = \frac{\pi}{\sqrt{2}}; \]

(b) \[ \int_0^\infty \frac{x \sqrt{x}}{(1 + x^3)^2} \, dx = \frac{\pi}{9}; \]

(c) \[ \int_{-\infty}^\infty \frac{e^{2x}}{(e^{3x} + 1)^2} \, dx = \frac{2\pi}{9\sqrt{3}}. \] (18 marks)
4 (i) In each of the following cases, find the function continuous on \([0, \infty)\) with the given Laplace transform:

(a) \(\frac{s - 1}{s(s + 1)}\) \((s > 0)\);
(b) \(\frac{4s + 2}{s^2 + 4}\) \((s > 0)\).

\(5\) marks

(ii) Express

\[
\frac{2(s + 5)(s^2 + 4)}{s^2(s^2 + 9)}
\]

in partial fractions. \(5\) marks

Suppose the functions \(f\) and \(g\) are continuous on \([0, \infty)\). Define the convolution \(f \ast g\) and state, without proof, a relation between \(L(f \ast g)\), \(L(f)\) and \(L(g)\). \(3\) marks

Using Laplace transforms, find the function \(f\) continuous on \([0, \infty)\) such that

\[
f'(t) + 5 \int_0^t f(u) \cos 2(t - u) \, du = 10 \quad (t \geq 0),
\]

and \(f\) satisfies the initial condition \(f(0) = 2\). \(12\) marks
5 (i) Let $b > 0$. Using Beta and Gamma functions show that

$$\int_0^\infty \frac{x^3}{x^6 + b^2} \, dx = \frac{\pi}{3\sqrt{3} b^{2/3}}.$$  

(7 marks)

Prove that

$$\int_0^\infty \sin(x^3) \, dx = \frac{\pi}{3\sqrt{3} \Gamma(2/3)}.$$  

(5 marks)

(ii) Using Laplace Transforms solve the differential equation

$$t \frac{d^2 y}{dt^2} + 3t \frac{dy}{dt} + (2t + 1)y = e^{-2t}$$

subject to the conditions $y(0) = 1$, $y(1) = \frac{2}{e^2}$.  

(13 marks)

End of Question Paper