Attempt all the questions. The allocation of marks is shown in brackets.
1  (i) Define the set $\text{Isom}_n$ of isometries of $\mathbb{R}^n$ and show that it is closed under composition and taking inverses. (7 marks)

(ii) Define $\psi : \text{Isom}_n \to O_n$ and show that it is a homomorphism. (3 marks)

(iii) The kernel of $\psi$ is given by the translations $T_a : x \mapsto x + a$ for $a \in \mathbb{R}^n$. Prove that for any $f \in \text{Isom}_n$ we have $fT_a f^{-1} = T_{\psi(f)a}$. (4 marks)

(iv) This part concerns plane isometries, i.e. $n = 2$.
   (a) Define the elements $R_\theta, S_\theta$ of $O_2$. (2 marks)
   (b) Give an explicit formula for $R_{\theta,a}$, the rotation by angle $\theta$ about $a \in \mathbb{R}^2$, and determine $\psi(R_{\theta,a})$. (2 marks)
   (c) Let $A$ be an element of $O_2$ with $\det(A) = -1$. Prove that $A = S_\theta$ for some $\theta$. (4 marks)
   (d) Let $f \in \text{Isom}_2$ with $\det(\psi(f)) = -1$. Show that $f^2$ is a translation. (3 marks)

2  (i) For any subgroup $H \leq \text{Isom}_2$ we defined its point group $\psi(H) \leq O_2$ and translation subgroup $\text{Trans}(H) \leq \mathbb{R}^2$. Explain which properties $\psi(H)$ and $\text{Trans}(H)$ need to satisfy for $H$ to be a wallpaper group. (3 marks)

(ii) Let $G$ be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)

(a) Describe geometrically all the translations, reflections and rotations (if any) in $G$. State clearly the vectors of any translations, lines of any reflections, and the centres and angles of any rotations. Specify one more element of $G$ that is not a translation, rotation or reflection. (8 marks)

(b) Find a list of three isometries that generate $G$. Justify your answer. (10 marks)

(c) Find $n$ and $\theta$ such that $\psi(G)$ is equal to $R_\theta D_n R_\theta^{-1}$. Justify your answer. (4 marks)
3 (i)  (a) Give the definition of the action of a group $G$ on a set $X$. 

\[ \text{(3 marks)} \]

(b) Given a group action explain how to define the corresponding map 
\[ \phi : G \rightarrow S(X) \]
and prove that it is a homomorphism taking values in $S(X)$. 

\[ \text{(5 marks)} \]

(ii) Let a group $G$ act on a set $X$.

(a) Show that this induces a group action of $G$ on the set of subsets of $X$ by the rule $g \ast N := gN = \{gn|n \in N\}$ for $N \subseteq X$. 

\[ \text{(4 marks)} \]

(b) If $G$ is a group with more than one element acting on itself by left multiplication then show that $g \ast N$ does not induce an action on the set of subgroups of $G$. 

\[ \text{(3 marks)} \]

(iii) Prove that the symmetry group of a regular tetrahedron centered at the origin is isomorphic to $S_4$. 

\[ \text{(6 marks)} \]

(iv) Describe a subset of $\mathbb{R}^2$ consisting of only two lines whose symmetry group is $D_4$. Write down all rotations and reflections preserving your subset. 

\[ \text{(4 marks)} \]

4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. 

\[ \text{(5 marks)} \]

(ii) Let $G$ be a group of order 91.

(a) Show that $G$ has a normal subgroup $N$ of order 13. 

\[ \text{(3 marks)} \]

(b) $G$ also has a normal subgroup $P$ of order 7 (no proof required). Prove that 

\[ pnp^{-1}n^{-1} = e \text{ for all } p \in P, n \in N. \]

\[ \text{(4 marks)} \]

(c) Define a function $\phi : P \times N \rightarrow G$ by $\phi(x, y) = xy$. Prove that $\phi$ is an isomorphism of groups. 

\[ \text{(6 marks)} \]

(iii) (a) Give the definition of a simple group. 

\[ \text{(2 marks)} \]

(b) By considering the conjugation action of $G$ on the set of Sylow 3-subgroups show that there is no simple group of order 36. 

\[ \text{(5 marks)} \]

End of Question Paper
Diagram for Question 2

Your registration number: ____________